

# SINGULAR HOCHSCHILD COHOMOLOGY AND ALGEBRAIC STRING OPERATIONS

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**ABSTRACT.** Given a differential graded (dg) symmetric Frobenius algebra  $A$  we construct an unbounded complex  $\mathcal{D}^*(A, A)$ , called the Tate-Hochschild complex, which arises as a totalization of a double complex having Hochschild chains as negative columns and Hochschild cochains as non-negative columns. We prove that the complex  $\mathcal{D}^*(A, A)$  computes the singular Hochschild cohomology of  $A$ . We construct a cyclic (or Calabi-Yau)  $A$ -infinity algebra structure, which extends the classical Hochschild cup and cap products, and an  $L$ -infinity algebra structure extending the classical Gerstenhaber bracket, on  $\mathcal{D}^*(A, A)$ . Moreover, we prove that the cohomology algebra  $H^*(\mathcal{D}^*(A, A))$  is a Batalin-Vilkovisky (BV) algebra with BV operator extending Connes' boundary operator. Finally, we show that if two Frobenius algebras are quasi-isomorphic as dg algebras then their Tate-Hochschild cohomologies are isomorphic and we use this invariance result to relate the Tate-Hochschild complex to string topology.

## 1. INTRODUCTION

For any differential graded (dg) associative algebra  $A$  over a ring  $\mathbb{K}$  such that  $A$  is projective as a  $\mathbb{K}$ -module, the Hochschild  $i$ -th cohomology group  $\mathrm{HH}^i(A, A)$  is defined as the group of morphisms from  $A$  to  $s^i A$  in  $\mathcal{D}(A \otimes_{\mathbb{K}} A^{\mathrm{op}})$ , the derived category of dg  $A$ - $A$ -bimodules, where  $s^i A$  is the dg  $A$ - $A$ -bimodule defined by  $(s^i A)^j = A^{i+j}$  with left and right actions induced by the multiplication of  $A$ .  $\mathrm{HH}^*(A, A)$  is a graded algebra with the Yoneda product. The *singular* Hochschild  $i$ -th cohomology group  $\mathrm{HH}_{\mathrm{sg}}^i(A, A)$  is defined as the group of morphisms from  $A$  to  $s^i A$  in the singular category  $\mathcal{D}_{\mathrm{sg}}(A \otimes_{\mathbb{K}} A^{\mathrm{op}})$ . The singular category  $\mathcal{D}_{\mathrm{sg}}(A \otimes_{\mathbb{K}} A^{\mathrm{op}})$  is the Verdier quotient (of triangulated categories) of  $\mathcal{D}^b(A \otimes_{\mathbb{K}} A^{\mathrm{op}})$ , the bounded derived category of finitely presented dg  $A$ - $A$ -bimodules, by the full sub-category  $\mathrm{Perf}(A \otimes_{\mathbb{K}} A^{\mathrm{op}})$  of  $\mathcal{D}^b(A \otimes_{\mathbb{K}} A^{\mathrm{op}})$  whose objects are perfect dg  $A$ - $A$ -bimodules [KoSo]. The singular category was defined by Buchweitz [Buc] and later rediscovered by Orlov [Orl].  $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$  is a graded algebra as well. In this article we discuss how the algebraic structure of the singular Hochschild cohomology of a dg symmetric Frobenius algebra relates to algebraic operations on Hochschild complexes which model constructions originated in string topology.

Let  $A$  be a dg associative algebra over a field  $\mathbb{K}$  together with a symmetric Frobenius pairing  $\langle \cdot, \cdot \rangle : A \otimes A \rightarrow \mathbb{K}$  of degree  $k \geq 0$ , meaning a symmetric pairing that induces a degree  $k$  isomorphism  $A \rightarrow A^\vee := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$  of dg  $A$ - $A$ -bimodules. For any such  $A$  we may construct a cochain complex, called the *Tate-Hochschild complex*, whose cohomology is isomorphic to the singular Hochschild cohomology of  $A$ . The Tate-Hochschild complex of  $A$  is a totalization of an unbounded double cochain complex denoted by  $\mathcal{D}^{*,*}(A, A)$  which may be constructed as follows: place the Hochschild chain complex of  $A$  on the side of the columns with negative degree (i.e.  $\mathcal{D}^{i,*}(A, A) := C_{-i-1,*}(A, A)$  for  $i < 0$ ), the Hochschild cochain complex of  $A$  on the side of the columns with non-negative degree (i.e.  $\mathcal{D}^{i,*}(A, A) := C^{i,*}(A, A)$  for  $i \geq 0$ ), and use the Frobenius structure on  $A$  to define a linear map  $\gamma : \mathcal{D}^{-1,*}(A, A) \rightarrow \mathcal{D}^{0,*}(A, A)$ , connecting the Hochschild chain complex and the Hochschild cochain complex, which extends the differentials defined on each side.

Here the  $C_{-m,p}(A, A)$  denotes the  $\mathbb{K}$ -module of Hochschild chains with monomial length  $m$  and total (homological) degree  $p$ , similarly for  $C^{m,p}(A, A)$  for Hochschild cochains. More precisely, the connecting map  $\gamma : \mathcal{D}^{-1,*}(A, A) \rightarrow \mathcal{D}^{0,*}(A, A)$  is the degree  $-k$  map given by the composition

$$(1) \quad \gamma : A \xrightarrow{\Delta} A \otimes A \xrightarrow{T} A \otimes A \xrightarrow{\mu} A,$$

where  $\mu : A \otimes A \rightarrow A$  is the product of the algebra  $A$ ,  $\Delta$  is the coproduct of  $A$  associated to the Frobenius pairing, namely

$$\Delta : A \xrightarrow{\cong} A^\vee \xrightarrow{\mu^\vee} A^\vee \otimes A^\vee \xrightarrow{\cong} A \otimes A,$$

and  $T$  is the braiding isomorphism

$$T(x \otimes y) := (-1)^{|x||y|} y \otimes x.$$

In order to get a total differential on  $\mathcal{D}^*(A, A) := \text{Tot}(\mathcal{D}^{*,*}(A, A))$  of degree  $+1$  we shift each of the negative columns by  $1 - k$ , where the totalization here means the direct sum totalization in the Hochschild chains direction and direct product totalization in the Hochschild cochains direction. A short calculation yields that  $\mathcal{D}^{*,*}(A, A)$  is in fact a double complex.

We describe explicitly a chain level lift of the graded associative algebra structure of  $\text{HH}_{\text{sg}}^*(A, A)$  to an  $A_\infty$ -algebra structure on  $\mathcal{D}^*(A, A)$ . The  $m_2$  operation of this  $A_\infty$ -algebra is a product  $\star : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^*(A, A)$  which extends the Hochschild cup product, defined for a pair of Hochschild cochains, and the Hochschild cap product, defined for a Hochschild cochain and a Hochschild chain satisfying certain degree constraints. We construct the  $A_\infty$ -algebra structure by first lifting the associative algebra structure of  $\text{HH}_{\text{sg}}^*(A, A)$  to a dg associative algebra on a chain complex  $\mathcal{C}_{\text{sg}}^*(A, A)$ , chain homotopy equivalent to  $\mathcal{D}^*(A, A)$  through a homotopy retraction, and then transferring such structure to a quasi-isomorphic  $A_\infty$ -algebra structure on  $\mathcal{D}^*(A, A)$ . Furthermore, we show the resulting  $A_\infty$ -algebra structure is cyclically compatible with a pairing on  $\mathcal{D}^{*,*}(A, A)$  induced by the Frobenius pairing of  $A$ . In other words, we construct a cyclic (or Calabi-Yau)  $A_\infty$ -algebra structure on the totalization of the double complex  $\mathcal{D}^{*,*}(A, A) = s^{1-k}C_{*,*}(A, A) \oplus C^{*,*}(A, A)$  which lifts the graded associative algebra structure of  $\text{HH}_{\text{sg}}^*(A, A)$ , where  $s^{1-k}$  denotes the degree shift by  $1 - k$ . We also describe an  $L_\infty$ -algebra structure on  $\mathcal{D}^{*,*}(A, A)$  extending the dg Lie algebra structure on Hochschild cochains with the Gerstenhaber bracket. Moreover, we prove that the product  $\star$  on  $\text{HH}_{\text{sg}}^*(A, A) \cong H^*(\mathcal{D}^*(A, A))$  is part of a BV-algebra structure. The BV operator may be defined on  $\mathcal{D}^*(A, A)$  and it extends Connes' boundary operator  $B$  on  $C_{*,*}(A, A)$ . We pose the following Deligne type conjecture: this BV-algebra can be lifted to an action of a dg operad, weakly equivalent to chains on the framed little disks operad, on  $\mathcal{D}^*(A, A)$ . In particular, the construction of such an action would extend the solution of the cyclic Deligne conjecture described by Kaufmann [Kau] and independently by Tradler-Zeinalian [TrZe].

It follows from a well known theorem of Jones that for any simply connected space  $X$  there is an isomorphism  $\text{HH}_*(C^*(X; \mathbb{Q}), C^*(X; \mathbb{Q})) \cong H^*(LX; \mathbb{Q})$  where  $LX$  is the free loop space on  $X$ . Moreover, for a  $k$ -dimensional simply connected closed manifold  $M$  we have an isomorphism of graded algebras  $\text{HH}^*(C^*(M; \mathbb{Q}), C^*(M; \mathbb{Q})) \cong s^k H_*(LM; \mathbb{Q})$ , where the left hand side is the Hochschild cohomology of the rational singular cochains on  $M$  with cup product and the right hand side is the singular homology of the free loop space of  $M$  with rational coefficients (shifted by  $k$ ) with the Chas-Sullivan loop product

[FeThVi]. The Chas-Sullivan loop product is an intersection type operation which combines the intersection product of the manifold  $M$  and concatenation of loops. The loop product led to the construction of more general *string topology operations* on chains of loops in a manifold by using the Poincaré duality of the underlying manifold and concatenating or deconcatenating loops according to different compatible patterns. String topology operations have counterparts in the Hochschild chain complex of a dg symmetric Frobenius algebra [TrZe, WaWe].

Since Hochschild homology is invariant under quasi-isomorphisms, it follows there is an isomorphism of graded vector spaces  $\mathrm{HH}_*(A, A) \cong H^*(LM)$  for any differential graded algebra  $A$  quasi-isomorphic to the rational singular cochains  $C^*(M; \mathbb{Q})$ . The singular cohomology with rational coefficients  $H^*(M; \mathbb{Q})$  is a graded symmetric Frobenius algebra equipped with the Poincaré duality pairing and the cup product, however, this pairing does not give a Frobenius structure at the level of singular cochains  $C^*(M; \mathbb{Q})$ . By the main result in [LaSt], for any simply connected manifold  $M$ , one may construct a commutative differential graded symmetric Frobenius algebra which is quasi-isomorphic, as a dg associative algebra, to  $C^*(M; \mathbb{Q})$  with the property that the induced isomorphism  $H^*(A) \cong H^*(C^*(M; \mathbb{Q}))$  preserves the Frobenius algebra structures. Using a Frobenius structure on a differential graded algebra  $A$  one may construct operations on the Hochschild chain complex  $C_{*,*}(A, A)$  analogue to string topology operations [WaWe, TrZe, Abb]. One of the main points we would like to stress in this paper is the following surprising observation: when the product  $\star : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^*(A, A)$  is applied to a pair of Hochschild chains we obtain an operation which was previously described in [Abb] and [Kla], in the case when  $A$  is commutative, and is believed to be intimately related to the Goresky-Hingston product on  $H^*(LM, M)$ , a string topology operation of degree  $k - 1$  constructed on the cohomology of the free loop space of a  $k$ -dimensional manifold relative to constant loops.

More precisely, for any symmetric Frobenius differential graded algebra  $A$  with pairing of degree  $k$ , Abbaspour describes in [Abb] an associative product  $\ast : C_{*,*}(A, A)^{\otimes 2} \rightarrow C_{*,*}(A, A)$  of degree  $k - 1$  which arises as a chain homotopy between two chain maps  $\ast_0, \ast_1 : C_{*,*}(A, A)^{\otimes 2} \rightarrow C_{*,*}(A, A)$  each of degree  $k$ . The operation  $\ast$  is sometimes called a *secondary* operation since it is a chain homotopy between two *primary* operations; note, in particular, that  $\ast$  is not a chain map. The construction of  $\ast$  resembles a chain level version of a geometric string topology product on  $C^*(LM; \mathbb{Q})$  (described in [GoHi] at the level of relative cohomology  $H^*(LM, M; \mathbb{Q})$ ) which is dual to an operation which, at the level of chains, considers self intersections in a chain of loops and splits loops at these intersection points. There are several ways of obtaining a chain map from  $\ast$ . In the case  $A$  is commutative Abbaspour suggests taking a subcomplex of  $C^{*,*}(A, A)$  and Klamt suggests modifying the product  $\ast$  in order to obtain a chain map. We do not assume commutativity for  $A$  and propose extending the chain complex  $C_{*,*}(A, A)$  by extending the differential through the map  $\gamma$  defined in (1). We may keep extending the resulting complex by the Hochschild cochains differential to obtain an unbounded double complex and we may extend the product  $\ast$  as well by Hochschild cup and cap products. The resulting complex, after appropriate degree shifts, is precisely  $\mathcal{D}^{*,*}(A, A)$  and the new product is  $\star$ . In conclusion, one may interpret  $\star : \mathcal{D}^{*,*}(A, A) \otimes \mathcal{D}^{*,*}(A, A) \rightarrow \mathcal{D}^{*,*}(A, A)$  as combination of the cup product in Hochschild cochains, which is an algebraic analogue of the Chas-Sullivan loop product, and the  $\ast$ -product in Hochschild chains, an algebraic analogue of the Goresky-Hingston product. We finish this article by describing the singular Hochschild cohomology of the dga of singular cochains on a simply connected manifold

$M$  in terms of the homology and cohomology of  $LM$ . We do this by first proving an invariance result for singular Hochschild cohomology under dga quasi-isomorphisms and then using a dg symmetric Frobenius algebra model for  $M$  as provided by the main result of [LaSt]. We use these results to calculate explicitly the singular Hochschild cohomology and its BV-algebra structure for the differential graded algebra  $C^*(S^n, \mathbb{Q})$  of singular cochains on spheres  $S^n$  for  $n > 1$ .

There is a close relationship between the singular Hochschild cohomology of the dga of singular cochains on a closed simply connected manifold  $M$  and the *Rabinowitz-Floer homology* of the unit cotangent bundle of  $M$  with its canonical symplectic structure as introduced in [CiFrOa]. The relationship comes from the isomorphism between the symplectic (co)homology of the unit cotangent bundle of  $M$  and the singular (co)homology of the free loop space of  $M$ . In fact, one should compare Theorem 1.10 in [CiFrOa] with Theorem 7.1 of this article. Moreover, in [CiOa] an algebra structure on the Rabinowitz-Floer side is constructed and we conjecture that such structure agrees with the algebra structure of the singular Hochschild cohomology of the dga of singular cochains of  $M$ . Based on the constructions of [CiOa], it is also evident that the singular Hochschild cohomology is closely related to the symplectic homology of the boundary of a Liouville domain. These topics will be explored in future research.

#### ACKNOWLEDGMENT

The first author acknowledges support by the ERC via the grant StG-259118-STEIN and the excellent working conditions at *Institut de Mathématiques de Jussieu-Paris Rive Gauche* (IMJ-PRG) where the first half of this project was completed and the support of University of Miami and Fordecyt-Conacyt during the second half of the project. We would like to thank Hossein Abbaspour, Ragnar Buchweitz, Tobias Dyckerhoff, Yizhi Huang, Dmitry Kaledin, Liang Kong, Alexandru Oancea, Bruno Vallette, Guodong Zhou and Alexander Zimmermann for stimulating discussions and comments. We would also like to thank Natalia Rodriguez for helping out with the diagrams and figures in this article.

## 2. PRELIMINARIES

In this section we recall some basic notions and constructions regarding differential graded (dg) algebras. For more details we refer the reader to [Abb, Kel1, Kel2].

### 2.1. Differential graded algebras.

**Definition 2.1.** Let  $\mathbb{K}$  be a commutative ring. A *differential graded (dg) algebra over  $\mathbb{K}$*  is a cochain complex  $(A^\bullet, d^\bullet)$  of  $\mathbb{K}$ -modules endowed with  $\mathbb{K}$ -linear maps  $\mu : A^n \times A^m \rightarrow A^{n+m}$ ,  $(a, b) \mapsto ab := \mu(a, b)$  such that  $d^{n+m}(ab) = d^n(a)b + (-1)^n ad^m(b)$  and such that  $\bigoplus_{n \in \mathbb{Z}} A^n$  becomes an associative  $\mathbb{K}$ -algebra with unit. Moreover, we say that a dg algebra  $A$  is *commutative* if

$$xy = (-1)^{\deg(x)\deg(y)}yx$$

for any homogeneous elements  $x, y \in A$ .

**Definition 2.2.** Let  $\mathbb{K}$  be a commutative ring. Let  $(A, d)$  be a dg algebra over  $\mathbb{K}$ . The *opposite dg algebra* is the dg algebra  $(A^{\text{op}}, d^{\text{op}})$  over  $\mathbb{K}$  where  $A^{\text{op}} := A$  as an  $\mathbb{K}$ -module,  $d^{\text{op}} = d$ , and multiplication is given by

$$a \cdot_{\text{op}} b = (-1)^{\deg(a)\deg(b)}ba$$

for homogeneous elements  $a, b \in A$ .

**Definition 2.3.** Let  $\mathbb{K}$  be a commutative ring. Let  $(A, d)$  and  $(B, d)$  be two differential graded algebras over  $\mathbb{K}$ . The *tensor product differential graded algebra* of  $A$  and  $B$  is the algebra  $A \otimes B$  with multiplication defined by

$$(a \otimes b)(a' \otimes b') := (-1)^{\deg(a') \deg(b)} aa' \otimes bb'$$

endowed with differential  $d_{A \otimes B}$  defined by the rule

$$d_{A \otimes B}(a \otimes b) = d(a) \otimes b + (-1)^{\deg(a)} a \otimes d(b).$$

**Definition 2.4.** Let  $\mathbb{K}$  be a commutative ring. Let  $(A, d)$  be a differential graded  $\mathbb{K}$ -algebra. A *(left) differential graded module*  $M$  over  $A$  is a left  $A$ -module  $M$  which has a grading  $M = \bigoplus_{n \in \mathbb{Z}} M^n$  and a differential  $d$  such that  $A^m M^n \subset M^{m+n}$ , such that  $d(M^n) \subset M^{n+1}$ , and such that

$$d(am) = d(a)m + (-1)^{\deg(a)} ad(m)$$

for any homogenous element  $a \in A$ .

**Definition 2.5.** Let  $(A, d)$  be a differential graded  $\mathbb{K}$ -algebra. Let  $(M, d)$  and  $(N, d)$  be two differential graded  $A$ -modules. We say that  $f : (M, d) \rightarrow (N, d)$  is a  $(A, d)$ -module morphism of degree  $n \in \mathbb{Z}$  if the following two conditions are satisfied,

- (1)  $f$  is a  $\mathbb{K}$ -linear map such that  $f(M^k) \subset N^{k+n}$  for any  $k \in \mathbb{Z}$ ,
- (2)  $f \circ d = (-1)^n d \circ f$  and  $f(ax) = (-1)^{\deg(a)n} af(x)$ , where  $a \in A$  is any homogenous element and  $x \in M$ .

**Remark 2.6.** We denote by  $(A, d)\text{-Mod}$  (or also by  $A\text{-Mod}$ , for short) the abelian category whose objects are left differential graded  $A$ -modules and morphisms are the  $(A, d)$ -module morphisms of degree zero. In particular, when  $A$  is an (ordinary) associative algebra, the category  $(A, d)\text{-Mod}$  is equivalent to the category of cochain complexes of  $A$ -modules. In the same manner, we may define the abelian category of *right differential graded  $(A, d)$ -modules*. If  $(M, d)$  is a left  $(A, d)$ -module, then we may think of  $(M, d)$  as a right  $(A^{\text{op}}, d^{\text{op}})$ -module with the action

$$m \cdot_{\text{op}} a = (-1)^{\deg(a) \deg(m)} am$$

for any homogenous elements  $a \in A$  and  $m \in M$ . For simplicity, we will call a morphism of  $(A, d)$ -modules of degree zero is called a morphism of  $(A, d)$ -modules.

For any differential graded  $\mathbb{K}$ -module  $(M, d_M)$ , define a new differential graded  $\mathbb{K}$ -module  $(sM, d_{sM})$ , as a graded  $\mathbb{K}$ -module  $(sM)^n := M^{n+1}$  and the differential  $d_{sM} := -d_M$ . Thus we have a *shift functor*  $s : (A, d)\text{-Mod} \rightarrow (A, d)\text{-Mod}$  sending a dg  $A$ -module  $M$  to  $sM$ . It is clear that the shift functor  $s$  is an equivalence and, moreover, it induces the shift functor of the homotopy category  $\mathcal{K}(A\text{-Mod})$  and derived category  $\mathcal{D}(A\text{-Mod})$  as triangulated categories (cf. [Kel1]).

Let  $\mathbb{K}$  be a commutative ring. Given two differential graded  $\mathbb{K}$ -modules  $(M, d)$  and  $(N, d)$ , we denote the switch of factors morphism  $T : M \otimes_{\mathbb{K}} N \rightarrow N \otimes_{\mathbb{K}} M$  by

$$T(x \otimes_{\mathbb{K}} y) := (-1)^{\deg(x) \deg(y)} y \otimes_{\mathbb{K}} x,$$

for any homogeneous elements  $x \in M$  and  $y \in N$ .

**Definition 2.7.** Let  $\mathbb{K}$  be a field. A *differential graded (dg) Frobenius  $\mathbb{K}$ -algebra* of degree  $k \geq 0$  is the data  $(A, d, \mu, \Delta)$  where

- (1)  $(A, d, \mu)$  is a differential graded unital and associative  $\mathbb{K}$ -algebra such that  $A$  is non-negatively graded (i.e.,  $A^{<0} = 0$ );
- (2)  $(A, \Delta)$  is a differential graded coassociative coalgebra of degree  $k$ . That is,  $\Delta : A \rightarrow A \otimes_{\mathbb{K}} A$  is a linear map of degree  $k$  such that

- (a)  $(\Delta \otimes \text{id})\Delta = (-1)^k(\text{id} \otimes \Delta)\Delta$
- (b)  $\Delta d = (-1)^k(d \otimes \text{id} + \text{id} \otimes d)\Delta$
- (3)  $\Delta : A \rightarrow A \otimes_{\mathbb{K}} A$  is a left and right differential graded  $A$ -module map (an  $A$ - $A$ -bimodule map)

We say  $(A, d, \mu, \Delta)$  is *counital* if it is equipped with a map of dg vector spaces  $\epsilon : A \rightarrow \mathbb{K}$  which is a counit for  $\Delta : A \rightarrow A \otimes A$ , i.e.  $(\text{id} \otimes \epsilon)\Delta = \text{id} = (\epsilon \otimes \text{id})\Delta$ . We say  $(A, d, \mu, \Delta)$  is *commutative* if  $(A, d, \mu)$  is a commutative dg algebra and *cocommutative* if  $T\Delta = (-1)^k\Delta$ . Let  $(A, d, \mu, \Delta)$  be a dg Frobenius  $\mathbb{K}$ -algebra of degree  $k$ . We say  $(A, d, \mu, \Delta)$  is a *dg symmetric Frobenius  $\mathbb{K}$ -algebra* if it is counital and if  $T\Delta(1) = (-1)^k\Delta(1)$ , where 1 is the unit of  $A$ . Note that a dg symmetric Frobenius  $\mathbb{K}$ -algebra is not necessarily commutative or cocommutative.

**Remark 2.8.** We give several remarks about a dg symmetric Frobenius algebra  $(A, d, \mu, \Delta, \epsilon)$  of degree  $k$ .

- (1) We will use generalized Sweedler's notation for the coproduct

$$\Delta(x) = \sum_{(x)} x' \otimes x'' = \sum x' \otimes x''.$$

Using this notation, a counit may be defined as a linear map  $\epsilon : A \rightarrow \mathbb{K}$  satisfying

$$x = \sum (-1)^{k \deg(x')} x' \epsilon(x'') = \sum \epsilon(x') x''.$$

We will also write  $\mu(xy) = xy$  for the product.

- (2) If we write  $\Delta(1) = \sum_i e_i \otimes f_i$ , then  $\deg(e_i) + \deg(f_i) = k$ . Since  $\Delta$  is a  $A$ - $A$ -bimodule homomorphism, then

$$\Delta(x) := \sum x' \otimes x'' = \sum_i e_i \otimes f_i x = \sum_i (-1)^{k \deg(x)} x e_i \otimes f_i.$$

- (3) A counit  $\epsilon : A \rightarrow \mathbb{K}$  defines a symmetric, invariant, non-degenerate inner product  $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$  by the formula  $\langle x, y \rangle := \epsilon(xy)$ . In particular,  $A$  is finite dimensional as a  $\mathbb{K}$ -vector space.
- (4) A commutative dg symmetric Frobenius algebra is cocommutative.

**Lemma 2.9.** *Let  $A$  be a dg symmetric Frobenius  $\mathbb{K}$ -algebra of degree  $k \geq 0$ . Then  $A^{>k} = 0$ .*

*Proof.* By Remark 2.8 (3) above, there is an  $A$ - $A$ -bimodule isomorphism of degree  $-k$  between  $A$  and  $\text{Hom}_{\mathbb{K}}(A, \mathbb{K})$ , where the degree of  $\text{Hom}_{\mathbb{K}}(A^i, \mathbb{K})$  is  $-i$ . We have also that  $A$  is finite dimensional, in particular,  $A$  is bounded. So we have that  $A^{>k} = 0$  and as  $\mathbb{K}$ -vector spaces,  $A^i \cong \text{Hom}_{\mathbb{K}}(A^{k-i}, \mathbb{K})$  for any  $i = 0, \dots, k$ .  $\square$

**2.2. Hochschild homology and cohomology.** Let  $\mathbb{K}$  be a commutative ring. Let  $(A, d)$  be a differential graded  $\mathbb{K}$ -algebra. Denote  $\overline{A} := A/(\mathbb{K} \cdot 1)$ . For any  $m \in \mathbb{Z}_{\geq 0}$ , we define differential graded  $A$ - $A$ -bimodules

$$\text{Bar}_{-m}(A) := A \otimes (s\overline{A})^{\otimes m} \otimes A.$$

For simplicity, we will drop the  $s$  indicating the shift writing an element  $a_0 \otimes s\overline{a_1} \otimes \dots \otimes s\overline{a_m} \otimes a_{m+1} \in A \otimes (s\overline{A})^{\otimes m} \otimes A$  as  $a_0 \otimes \overline{a_1} \otimes \dots \otimes \overline{a_m} \otimes a_{m+1}$ . Thus  $\deg(a_0 \otimes \overline{a_1} \otimes \dots \otimes \overline{a_m} \otimes a_{m+1}) = \sum_{i=0}^{m+1} \deg(\overline{a_i}) - m$ . Hence, an element  $\overline{a}$  should always be interpreted as  $s\overline{a}$ , namely, as the class in  $\overline{A}$  represented by  $a \in A$  with a degree shift indicated by  $s$ .

For each  $m > 0$  we have a morphism (of degree one) of dg  $A$ - $A$ -bimodules

$$(2) \quad b_{-m} : \text{Bar}_{-m}(A) \rightarrow \text{Bar}_{-m+1}(A)$$

defined by sending  $a_0 \otimes \overline{a_1} \otimes \cdots \otimes a_{m+1} \in \text{Bar}_{-m}(A)$  to

$$\begin{aligned} & a_0 a_1 \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} + \\ & \sum_{i=1}^{m-1} (-1)^{\epsilon_i} a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} + \\ & (-1)^{\epsilon_m} a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m-1}} \otimes a_m a_{m+1}, \end{aligned}$$

where  $\epsilon_i := \sum_{j=0}^{i-1} \deg(a_j) - i$ ; and we define  $b_0 : \text{Bar}_0(A) = A \otimes A \rightarrow A$  to be the multiplication  $\mu$  of  $A$ . As pointed above, we have dropped the shift functor  $s$  and we have written  $a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1}$  for the element  $a_0 \otimes s\overline{a_1} \otimes \cdots \otimes s\overline{a_{i-1}} \otimes s(\overline{a_i a_{i+1}}) \otimes s\overline{a_{i+2}} \otimes \cdots \otimes s\overline{a_m} \otimes a_{m+1}$ . We will use this convention throughout the rest of the paper.

It is straightforward to verify that we obtain a well defined map  $b = b_* : \text{Bar}_*(A) \rightarrow \text{Bar}_{*+1}(A)$  satisfying  $b \circ b = 0$ . Note that  $s^{-m} b_{-m} : s^{-m} \text{Bar}_{-m}(A) \rightarrow s^{-m+1} \text{Bar}_{-m+1}(A)$  is a degree zero morphisms of dg  $A$ - $A$ -bimodules, hence we obtain a complex of dg  $A$ - $A$ -bimodules:

$$(3) \quad \text{Bar}_*(A) : \cdots \longrightarrow s^{-m} \text{Bar}_{-m}(A) \xrightarrow{s^{-m} b_{-m}} s^{-m+1} \text{Bar}_{-m+1}(A) \longrightarrow \cdots \xrightarrow{b_0} A \longrightarrow 0$$

**Lemma 2.10.** *The complex  $\text{Bar}_*(A)$  is exact in the category of dg  $A$ - $A$ -bimodules.*

*Proof.* Define  $s_m : s^{-m} \text{Bar}_{-m}(A) \rightarrow s^{-m-1} \text{Bar}_{-m-1}(A)$  to be the map which sends

$$a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} \in \text{Bar}_{-m}(A)$$

to

$$a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}} \otimes 1 \in \text{Bar}_{-m-1}(A).$$

We then have  $s \circ b + b \circ s = \text{id}_{\text{Bar}_*(A)}$ , as desired.  $\square$

**Remark 2.11.** Recall that the *two-sided bar construction* of a differential graded  $\mathbb{K}$ -algebra  $A$  is given by the differential graded  $A$ - $A$ -bimodule  $B(A, A, A) := A \otimes T(s\overline{A}) \otimes A$  with the differential  $d = d_0 + d_1$ , where  $d_0$  is the internal differential defined by the following rule:

$$\begin{aligned} d_0(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1}) = & d(a_0) \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} - \\ & \sum_{i=1}^m (-1)^{\epsilon_i} a_0 \otimes \overline{a_1} \otimes \overline{d(a_i)} \otimes \overline{a_{i+1}} \otimes \cdots \otimes a_{m+1} + \\ & (-1)^{\epsilon_{m+1}} a_0 \otimes \overline{a_1} \otimes \cdots \otimes d(a_{m+1}) \end{aligned}$$

and the differential  $d_1$  is the external differential given by  $b$  defined in (2). In fact, the two-sided bar construction  $B(A, A, A)$  above is initially constructed from the following double complex  $\text{Bar}_{*,*}(A)$  whose  $(m, p)$ -term is

$$\text{Bar}_{-m,p}(A) := (A \otimes (s\overline{A})^{\otimes m} \otimes A)^p.$$

That is,  $a_0 \otimes \overline{a_1} \otimes \cdots \otimes a_{m+1} \in \text{Bar}_{-m,p}(A)$  if

$$\sum_{i=0}^{m+1} \deg(a_i) - m = p.$$

The vertical differential  $d_v : \text{Bar}_{*,p}(A) \rightarrow \text{Bar}_{*,p+1}(A)$  is exactly  $d_0$  defined above and the horizontal differential  $d_h : \text{Bar}_{-m,*}(s\overline{A}) \rightarrow \text{Bar}_{-m+1,*}(s\overline{A})$  is defined to be  $d_1$ . From Lemma 2.10, it follows that the canonical morphism  $\pi : B(A, A, A) \rightarrow A$  is a quasi-isomorphism of dg  $A$ - $A$ -bimodules, hence  $B(A, A, A)$  is a cofibrant resolution of  $A$  (cf.

[Kel2, Section 3.2]). In particular, if  $A$  is an (ordinary) associative  $\mathbb{K}$ -algebra such that  $A$  is projective as a  $\mathbb{K}$ -module, then  $B(A, A, A)$  is a projective resolution of  $A$  as an  $A$ - $A$ -bimodule.

For any  $n \in \mathbb{Z}$ , the *Hochschild cohomology*  $\mathrm{HH}^n(A, A)$  of degree  $n$  of a differential graded algebra  $A$  is defined as the Hom-space  $\mathrm{Hom}_{\mathcal{D}(A \otimes A^{\mathrm{op}})}(A, s^n A)$  in the derived category  $\mathcal{D}(A \otimes A^{\mathrm{op}})$ . Since  $B(A, A, A)$  is a cofibrant resolution of  $A$  as an  $A$ - $A$ -bimodule, we have that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}(A \otimes A^{\mathrm{op}})}(A, s^n A) &\cong \mathrm{Hom}_{\mathcal{K}(A \otimes A^{\mathrm{op}})}(B(A, A, A), s^n A) \\ &\cong H_n(R \mathrm{Hom}_{\mathbb{K}}(T(\overline{sA}), A)). \end{aligned}$$

Note that  $R \mathrm{Hom}_{\mathbb{K}}(T(\overline{sA}), A)$  is the (product) total complex of the following double complex  $C^{*,*}(A, A)$  located in the right half plane,

$$(4) \quad \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A^1 & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}(\overline{sA}, A)^2 & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}((\overline{sA})^{\otimes 2}, A)^3 \longrightarrow \dots \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A^0 & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}(\overline{sA}, A)^1 & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}((\overline{sA})^{\otimes 2}, A)^2 \longrightarrow \dots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}(\overline{sA}, A)^0 & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}((\overline{sA})^{\otimes 2}, A)^1 \longrightarrow \dots \\ & & & & \uparrow & & \uparrow \\ & & & & \vdots & & \vdots \end{array}$$

where  $C^{m,p}(A, A) := \mathrm{Hom}_{\mathbb{K}}((\overline{sA})^{\otimes m}, A)^p$  is the set of morphisms  $f : (\overline{sA})^{\otimes m} \rightarrow A$  of degree  $p$  and the differentials are induced from the ones of  $\mathrm{Bar}_{*,*}(\overline{sA})$  (cf. Remark 2.11). More precisely, the vertical differential  $\delta^v : \mathrm{Hom}_{\mathbb{K}}((\overline{sA})^{\otimes m}, A)^p \rightarrow \mathrm{Hom}_{\mathbb{K}}((\overline{sA})^{\otimes m}, A)^{p+1}$  is given by

$$\begin{aligned} \delta^v(f)(\overline{a_1} \otimes \dots \otimes \overline{a_m}) &:= df(\overline{a_1} \otimes \dots \otimes \overline{a_m}) + \\ &\quad \sum_{i=1}^m (-1)^{\epsilon_i} f(\overline{a_1} \otimes \dots \otimes \overline{da_i} \otimes \overline{a_{i+1}} \otimes \dots \otimes \overline{a_m}), \end{aligned}$$

where  $\epsilon_i := p + i - 1 + \sum_{j=1}^{i-1} \deg(a_j)$  and the horizontal differential  $\delta^h : \mathrm{Hom}_{\mathbb{K}}((\overline{sA})^{\otimes m}, A)^p \rightarrow \mathrm{Hom}_{\mathbb{K}}((\overline{sA})^{\otimes m+1}, A)^{p+1}$  is given by

$$\begin{aligned} \delta^h(f)(\overline{a_1} \otimes \dots \otimes \overline{a_{m+1}}) &:= (-1)^{\deg(a_1)p} a_1 f(\overline{a_2} \otimes \dots \otimes \overline{a_{m+1}}) + \\ &\quad \sum_{i=1}^m (-1)^{\epsilon_i} f(\overline{a_1} \otimes \dots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_i} \otimes \dots \otimes \overline{a_{m+1}}) - \\ &\quad (-1)^{\epsilon_{m+1}} f(\overline{a_1} \otimes \dots \otimes \overline{a_m}) a_{m+1}. \end{aligned}$$

Define  $(C^*(A, A), \delta) = \mathrm{Tot}^{\Pi}(C^{*,*}(A, A))$ , so  $C^n(A, A) = \prod_{p \in \mathbb{Z}_{\geq 0}} C^{p,n}(A, A)$ . We call  $(C^*(A, A), \delta)$  the Hochschild cochain complex of  $A$ . Clearly,  $H^n(\tilde{C}^*(A, A)) \cong \mathrm{HH}^n(A, A)$  for any  $n \in \mathbb{Z}$ .

For any  $n \in \mathbb{Z}$ , the *Hochschild homology*  $\mathrm{HH}_n(A, A)$  of degree  $n$  of a differential graded



algebra  $A$  is defined as the  $n$ -th homology group of the derived tensor product  $A \otimes_{A \otimes A^{\text{op}}}^{\mathbb{L}} A$ . Using the fact that  $B(A, A, A)$  is a cofibrant resolution, we obtain that

$$\begin{aligned} \text{HH}_n(A, A) &\cong H_n(B(A, A, A) \otimes_{A \otimes A^{\text{op}}} A) \\ &\cong H_n(T(s\bar{A})) \otimes_{\mathbb{K}} A \end{aligned}$$

Similarly, we note that  $T(s\bar{A}) \otimes_{\mathbb{K}} A$  is the (direct sum) total complex of the following double complex  $C_{*,*}(A, A)$  located in the left half plane,

$$(5) \quad \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & ((s\bar{A})^{\otimes 3} \otimes A)^2 & \longrightarrow & ((s\bar{A})^{\otimes 2} \otimes A)^3 & \longrightarrow & (s\bar{A} \otimes A)^4 & \longrightarrow & A^5 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & ((s\bar{A})^{\otimes 3} \otimes A)^1 & \longrightarrow & ((s\bar{A})^{\otimes 2} \otimes A)^2 & \longrightarrow & (s\bar{A} \otimes A)^3 & \longrightarrow & A^4 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & ((s\bar{A})^{\otimes 3} \otimes A)^0 & \longrightarrow & ((s\bar{A})^{\otimes 2} \otimes A)^1 & \longrightarrow & (s\bar{A} \otimes A)^2 & \longrightarrow & A^3 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

where  $C_{-m,p}(A, A) := ((s\bar{A})^{\otimes m} \otimes A)^p$  is the set of the elements of degree  $p$  in  $(s\bar{A})^{\otimes m} \otimes A$ . The vertical differential  $\delta^v : ((s\bar{A})^{\otimes m} \otimes A)^p \rightarrow ((s\bar{A})^{\otimes m} \otimes A)^{p+1}$  is given by

$$\begin{aligned} \delta^v(\bar{a}_1 \otimes \cdots \otimes \bar{a}_m \otimes a_{m+1}) &:= d(\bar{a}_1) \otimes \bar{a}_2 \otimes \cdots \otimes a_{m+1} + \\ &\quad \sum_{i=1}^m (-1)^{\epsilon_i} \bar{a}_1 \otimes \cdots \otimes \bar{a}_{i-1} \otimes d(\bar{a}_i) \otimes \bar{a}_{i+1} \otimes \cdots \otimes a_{m+1} \end{aligned}$$

where  $\epsilon_i := \sum_{j=0}^{i-1} \deg(a_j) + i - 1$  and the horizontal differential  $\delta^h : (A \otimes (s\bar{A})^{\otimes m})^p \rightarrow (A \otimes (s\bar{A})^{\otimes m-1})^{p-1}$  is given by

$$\begin{aligned} \delta^h(\bar{a}_1 \otimes \cdots \otimes \bar{a}_m \otimes a_{m+1}) &:= \sum_{i=1}^m (-1)^{\epsilon_i} \bar{a}_1 \otimes \cdots \otimes \bar{a}_i \bar{a}_{i+1} \otimes \bar{a}_{i+2} \otimes \cdots \otimes a_{m+1} + \\ &\quad (-1)^{\epsilon_{m+1}} \bar{a}_1 \otimes \cdots \otimes \bar{a}_{m-1} \otimes a_m a_{m+1} + \\ &\quad \bar{a}_2 \otimes \cdots \otimes \bar{a}_m \otimes a_{m+1} a_1. \end{aligned}$$

Define  $(C_*(A, A), \delta) := \text{Tot}^{\oplus}(C_{*,*}(A, A))$ , so  $C_n(A, A) = \bigoplus_{p \in \mathbb{Z}_{\geq 0}} C_{-p,n}(A, A)$ . We call  $(C_*(A, A), \delta)$  the Hochschild chain complex of  $A$ . Note that  $C_n(A, A)$  might be non zero even for  $n$  a negative integer. Also note that in our convention  $\deg \delta = +1$  even if we call  $C_n(A, A)$  a *chain* complex. Clearly,  $H_n(C_*(A, A)) \cong \text{HH}_n(A, A)$  for any  $n \in \mathbb{Z}$ .

**Remark 2.12.** For any dg  $A$ - $A$ -bimodule  $M$ , the Hochschild cohomology  $\text{HH}^*(A, M)$  and homology  $\text{HH}_*(A, M)$  can be defined in a similar way. They are computed by the Hochschild cochain complex  $(C^*(A, M), \delta)$  and the Hochschild chain complex  $(C_*(A, M), \delta)$  with coefficients in  $M$  constructed similarly as above.

### 3. SINGULAR HOCHSCHILD COHOMOLOGY

In this section we define the singular Hochschild cohomology of a dg associative algebra  $A$ . We continue by defining the Tate-Hochschild complex  $\mathcal{D}^*(A, A)$  for a dg symmetric

Frobenius algebra  $A$  and show that  $\mathcal{D}^*(A, A)$  computes the singular Hochschild cohomology of  $A$ . Finally, for any dg associative algebra  $A$  we describe its singular Hochschild complex  $\mathcal{C}_{\text{sg}}^*(A, A)$  and show that, if  $A$  is a dg symmetric Frobenius algebra, then  $\mathcal{C}_{\text{sg}}^*(A, A)$  and  $\mathcal{D}^*(A, A)$  are chain homotopy equivalent through a homotopy retraction.

**3.1. The definition of singular Hochschild cohomology.** Throughout this subsection we fix a field  $\mathbb{K}$  and a dg  $\mathbb{K}$ -algebra  $A$  concentrated on non-negative degrees.

**Definition 3.1.** The *singular category*  $\mathcal{D}_{\text{sg}}(A)$  of  $A$  is defined as the Verdier quotient of triangulated categories  $\mathcal{D}^b(A\text{-mod})/\text{Perf}(A)$ , where  $\mathcal{D}^b(A\text{-mod})$  is the bounded derived category of finitely presented left dg  $A$ -modules and  $\text{Perf}(A)$  is the full sub-category of  $\mathcal{D}^b(A\text{-mod})$  whose objects are perfect dg  $A$ -modules ([KoSo]).

**Definition 3.2.** Let us denote  $A\text{-Mod}_{\text{inj}}$  the full sub-category of  $A\text{-Mod}$  consisting of the objects whose underlying graded  $A^\dagger$ -modules are injective, where  $A^\dagger$  denotes the underlying graded algebra of  $(A, d)$  obtained by forgetting the differential  $d$ . Let  $\mathcal{K}(A\text{-Mod}_{\text{inj}})$  be the homotopy category of  $A\text{-Mod}_{\text{inj}}$ ,  $\mathcal{K}_{ac}(A\text{-Mod}_{\text{inj}})$  the full sub-category of  $\mathcal{K}(A\text{-Mod}_{\text{inj}})$  consisting of objects  $C$  which are acyclic (i.e.,  $H^*(C) = 0$ ), and  $\mathcal{K}_{ac}(A \otimes A^{\text{op}}\text{-Mod}_{\text{inj}})^c$  the full sub-category of compact objects.

**Theorem 3.3** (Corollary 5.4.[Kra], Corollary 2.2.2.[Bec]). *The full-subcategory of  $\mathcal{K}_{ac}(A\text{-Mod}_{\text{inj}})$  consisting of compact objects is equivalent (up to direct summands) to the singular category  $\mathcal{D}_{\text{sg}}(A)$ . That is, we have an equivalence of triangulated categories*

$$S : \mathcal{D}_{\text{sg}}(A) \xrightarrow{\cong} (\mathcal{K}_{ac}(A\text{-Mod}_{\text{inj}}))^c.$$

where  $S$  is the stabilization functor (cf. [Kra, Corollary 5.4.]).

*Proof.* Recall from [Bec, Corollary 2.2.2.] that we have the following recollement of triangulated categories.

$$\begin{array}{ccccc} & I_\rho & & Q_\rho & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{K}_{ac}(A\text{-Mod}_{\text{inj}}) & \xrightarrow{I} & \mathcal{K}(A\text{-Mod}_{\text{inj}}) & \xrightarrow{Q} & \mathcal{D}(A) \\ & \curvearrowleft & & \curvearrowright & \\ & I_\lambda & & Q_\lambda & \end{array}$$

It follows from [Nee, Theorem 2.1.] that

$$I_\rho : (\mathcal{K}(A\text{-Mod}_{\text{inj}}))^c / (\mathcal{D}(A))^c \rightarrow (\mathcal{K}_{ac}(A\text{-Mod}_{\text{inj}}))^c$$

is fully-faithful and moreover the idempotent completion  $((\mathcal{K}(A\text{-Mod}_{\text{inj}}))^c / (\mathcal{D}(A))^c)^\omega$  is equivalent to  $(\mathcal{K}_{ac}(A\text{-Mod}_{\text{inj}}))^c$ . On the other hand, we note that

$$(\mathcal{K}(A\text{-Mod}_{\text{inj}}))^c \cong \mathcal{D}^b(A\text{-mod})$$

and

$$\mathcal{D}(A)^c \cong \text{Perf}(A).$$

So  $I_\rho : \mathcal{D}_{\text{sg}}(A) \rightarrow (\mathcal{K}_{ac}(A\text{-Mod}_{\text{inj}}))^c$  is an equivalence (up to direct summands) of triangulated categories.  $\square$

**Definition 3.4.** Let  $A$  be a dg  $\mathbb{K}$ -algebra concentrated on non-negative degrees such that  $H^i(A) = 0$  for  $i \gg 0$ . Suppose that  $A$  is projective as a  $\mathbb{K}$ -module. The *singular Hochschild cohomology group* of degree  $i$  is defined as

$$\text{HH}_{\text{sg}}^i(A, A) := \text{Hom}_{\mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})}(A, s^i A)$$

for any  $i \in \mathbb{Z}$ .

**Remark 3.5.** From Theorem 3.3, it follows that for any  $i \in \mathbb{Z}$ ,

$$\mathrm{HH}_{\mathrm{sg}}^i(A, A) \cong \mathrm{Hom}_{\mathcal{K}_{ac}}(A \otimes A^{\mathrm{op}\text{-}\mathrm{Mod}_{\mathrm{inj}}})^c(S(A), s^i S(A)).$$

The rest of this section is devoted to constructing two different complexes the homologies of which are both isomorphic to the singular Hochschild cohomology  $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$  in the case of a dg symmetric Frobenius  $\mathbb{K}$ -algebra  $A$  of degree  $k \geq 0$ .

**3.2. The Tate-Hochschild complex of a dg symmetric Frobenius algebra.** In this subsection we fix a field  $\mathbb{K}$  and a dg symmetric Frobenius  $\mathbb{K}$ -algebra  $A$  of degree  $k$ .

Let  $(C, d, \Delta, \epsilon)$  be a counital dg coalgebra of degree zero. Denote  $\overline{C} := \mathrm{Ker}(\epsilon)$ . Define the *two-sided cobar construction*  $\Omega(C, C, C)$  of  $C$  to be  $C \otimes T(s^{-1}\overline{C}) \otimes C$  with the differential  $d = d_0 + d_1$ , where  $d_0$  is the internal differential given by

$$\begin{aligned} d_0(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1}) &= d(a_0) \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} - \\ &\quad \sum_{i=1}^m (-1)^{\epsilon_i} a_0 \otimes \overline{a_1} \otimes \cdots \otimes d(\overline{a_i}) \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} + \\ &\quad (-1)^{\epsilon_{m+1}} a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes d(a_{m+1}) \end{aligned}$$

and the differential  $d_1$  is the external differential given by

$$\begin{aligned} d_1(a_0 \otimes \overline{a_1} \otimes \cdots \otimes a_{m+1}) &:= \Delta(a_0) \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} + \\ &\quad \sum_{i=1}^m (-1)^{\epsilon_{i-1}} a_0 \otimes \overline{a_1} \otimes \cdots \otimes \Delta(\overline{a_i}) \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} - \\ &\quad (-1)^{\epsilon_m} a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes \Delta(a_{m+1}) \end{aligned}$$

where  $\epsilon_i := \sum_{j=0}^i \deg(a_j) + i$ . Note that the range of  $\Delta$  is  $C \otimes C$ , however, from the following lemma it follows that  $d_1$  is well-defined as a map  $C \otimes T(s^{-1}\overline{C}) \otimes C \rightarrow C \otimes T(s^{-1}\overline{C}) \otimes C$ .

**Lemma 3.6.** *Let  $x := a_0 \otimes \overline{a_1} \otimes \cdots \otimes a_{m+1}$  be an element in  $C \otimes (s^{-1}\overline{C})^{\otimes m} \otimes C$ . Then we have  $d_1(x) \in C \otimes (s^{-1}\overline{C})^{\otimes m+1} \otimes C$ .*

*Proof.* It is sufficient to show that  $(\mathrm{id}^{\otimes i} \otimes \epsilon \otimes \mathrm{id}^{\otimes m+1-i})(d_1(x)) = 0$  for any  $0 < i < m$ . Indeed, we have

$$\begin{aligned} (\mathrm{id}^{\otimes i} \otimes \epsilon \otimes \mathrm{id}^{\otimes m+1-i})(d_1(x)) &= (-1)^{\epsilon_{i-1}} a_0 \otimes \overline{a_1} \otimes \cdots \otimes (\epsilon \otimes \mathrm{id})\Delta(a_i) \otimes \cdots \otimes a_{m+1} + \\ &\quad (-1)^{\epsilon_{i-2}} a_0 \otimes \overline{a_1} \otimes \cdots \otimes (\mathrm{id} \otimes \epsilon)\Delta(a_{i-1}) \otimes \cdots \otimes a_{m+1} \\ &= 0, \end{aligned}$$

where the first identity follows from the fact that  $\epsilon(\overline{a_i}) = 0$  for any  $i = 1, \dots, m$  and the second identity holds since  $(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta$ .  $\square$

**Remark 3.7.** Analogous to the two-sided bar construction, the two-sided cobar construction  $\Omega(C, C, C)$  is the total complex of the double complex whose  $(m, p)$ -term is  $(C \otimes (s^{-1}\overline{C})^{\otimes m} \otimes C)^p$  with the horizontal differential  $d_1$  and the vertical differential  $d_0$ .

We have a morphism of dg  $\mathbb{K}$ -modules  $\iota : C \rightarrow \Omega(C, C, C)$  which is induced by the coproduct  $\Delta : C \rightarrow C \otimes C$ .

**Lemma 3.8.**  *$\iota : C \rightarrow \Omega(C, C, C)$  is a quasi-isomorphism of dg  $\mathbb{K}$ -modules.*

*Proof.* It is sufficient to show that each row of the extended double complex is exact. Namely, for each  $p \in \mathbb{Z}$ , the following complex is exact.

$$0 \longrightarrow C^p \xrightarrow{\Delta} (C \otimes C)^p \xrightarrow{d_1} \dots \longrightarrow (C \otimes (s^{-1}\overline{C})^{\otimes m} \otimes C)^{p+m} \longrightarrow \dots$$

Let us construct a homotopy  $s_m : (C \otimes (s^{-1}\overline{C})^{\otimes m} \otimes C)^{p+m} \rightarrow (C \otimes (s^{-1}\overline{C})^{\otimes m-1} \otimes C)^{p+m-1}$  which sends  $a_0 \otimes \overline{a_1} \otimes \dots \otimes a_{m+1}$  to

$$(-1)^{\epsilon_m} a_0 \otimes \overline{a_1} \otimes \dots \otimes a_m \epsilon(a_{m+1}) \in (C \otimes (s^{-1}\overline{C})^{\otimes m-1} \otimes C)^{p+m-1}$$

where  $\epsilon_m := \sum_{i=1}^m \deg(a_i) + m$ . It is straightforward to check that  $s \circ d_1 + d_1 \circ s = \text{id}$ . Therefore the mapping cone  $\text{cone}(\iota)$  is acyclic and thus  $\iota$  is a quasi-isomorphism.  $\square$

Let us go back to the case where  $A$  is a dg symmetric Frobenius  $\mathbb{K}$ -algebra of degree  $k$ . Then  $(s^k A, s^k \Delta, s^k \epsilon)$  is a counital dg coalgebra of degree zero, where

$$s^k \Delta : s^k A \xrightarrow{s^k \Delta} s^{2k}(A \otimes A) \xrightarrow{\cong} s^k A \otimes s^k A$$

and

$$s^k \epsilon : s^k A \xrightarrow{s^k \epsilon} s^k s^{-k} \mathbb{K} \xrightarrow{\cong} \mathbb{K}.$$

To simplify the notation, we denote  $(s^k A, s^k \Delta, s^k \epsilon)$  by  $(C, \Delta, \epsilon)$ . It follows from Lemma 3.8 and the fact that  $\Delta : C \rightarrow C \otimes C$  is a morphism of dg  $A$ - $A$ -bimodules, that the two-sided cobar construction  $\Omega(C, C, C)$  is quasi-isomorphic to  $C$  as a dg  $A$ - $A$ -bimodule.

We have a morphism of dg  $A$ - $A$ -bimodules  $\tau : B(A, A, A) \rightarrow s^{-k} \Omega(C, C, C)$  which is, by definition, the composition of the following morphisms

$$(6) \quad \tau : B(A, A, A) \xrightarrow{\pi} A \xrightarrow{\cong} s^{-k} C \xrightarrow{s^{-k} \iota} s^{-k} \Omega(C, C, C)$$

where  $\pi$  is the composition  $B(A, A, A) \twoheadrightarrow A \otimes A \xrightarrow{\mu} A$  and  $\iota$  is the composition  $C \xrightarrow{\Delta} C \otimes C \hookrightarrow \Omega(C, C, C)$ . Since  $\pi$  and  $\iota$  are both quasi-isomorphisms, so is the morphism  $\tau$ . Thus the mapping cone  $\text{cone}(\tau)$  is acyclic, so  $\text{cone}(\tau) \in \mathcal{K}_{ac}(A \otimes A^{\text{op}}\text{-Mod}_{\text{inj}})$ .

**Lemma 3.9.** *We have an isomorphism*

$$S(A) \cong \text{cone}(\tau)$$

in  $\mathcal{K}_{ac}(A \otimes A^{\text{op}}\text{-Mod}_{\text{inj}})^c$ , where  $S$  is the stabilization functor from  $\mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})$  to  $\mathcal{K}_{ac}(A \otimes A^{\text{op}}\text{-Mod}_{\text{inj}})^c$  (cf. Theorem 3.3).

**Remark 3.10.** The mapping cone  $\text{cone}(\tau)$  can be viewed as the total complex of the double complex  $\mathcal{E}(A, A, A)$  whose  $(p, q)$ -term is defined to be  $(s^{-k}(C \otimes (s^{-1}\overline{C})^{\otimes p-1} \otimes C))^{p+q-1}$  when  $p > 0$  and  $(A \otimes (s\overline{A})^{\otimes -p} \otimes A)^{p+q}$  when  $p \leq 0$ . Roughly speaking, the double complex is obtained by connecting  $B(A, A, A)$  and  $s^{-k} \Omega(C, C, C)$  via the morphism  $\Delta \circ \mu$ , namely

$$\dots \rightarrow A \otimes (s\overline{A})^{\otimes 2} \otimes A \rightarrow A \otimes s\overline{A} \otimes A \rightarrow A \otimes A \xrightarrow{\Delta \circ \mu} s^{-1-k}(C \otimes C) \rightarrow s^{-1-k}(C \otimes s^{-1}\overline{C} \otimes C) \rightarrow \dots$$

Let us consider the differential graded Hom-space

$$\text{Hom}_{A \otimes A^{\text{op}}}(\text{cone}(\tau), A) = \text{Hom}_{A \otimes A^{\text{op}}}(\mathcal{E}(A, A, A), A).$$

By base change, we obtain a differential graded  $\mathbb{K}$ -module  $\mathcal{D}^*(A, A)$ , where

$$\mathcal{D}^n(A, A) := \prod_{p \in \mathbb{Z}_{\geq 0}} \text{Hom}_{\mathbb{K}}((s\overline{A})^{\otimes p}, A)^n \oplus \bigoplus_{p \in \mathbb{Z}_{\geq 0}} \text{Hom}_{\mathbb{K}}((s^{-1}\overline{C})^{\otimes p}, A)^{n-k+1}$$

with the differential  $\delta := \delta_0 + \delta_1$ , where  $\delta_0$  is the internal differential given by, for  $f \in \text{Hom}_{\mathbb{K}}((s\bar{A})^{\otimes p}, A)^n$  or  $f \in \text{Hom}_{\mathbb{K}}((s^{-1}\bar{C})^{\otimes p}, A)^{n-k+1}$ ,

$$\delta_0(f)(\bar{a}_1 \otimes \cdots \otimes \bar{a}_p) = d(f(\bar{a}_1 \otimes \cdots \otimes \bar{a}_p)) + \sum_{i=1}^p (-1)^{\epsilon_i} f(\bar{a}_1 \otimes \cdots \otimes \overline{da_i} \otimes \cdots \otimes \bar{a}_p).$$

$\delta_1$  is the external differential given by

(1) for  $f \in \text{Hom}_{\mathbb{K}}((s\bar{A})^{\otimes p}, A)^n$

$$\begin{aligned} \delta_1(f)(\bar{a}_1 \otimes \cdots \otimes \bar{a}_{p+1}) := & a_1 f(\bar{a}_2 \otimes \cdots \otimes \bar{a}_{p+1}) + \\ & \sum_{i=1}^p (-1)^{\epsilon_i} f(\bar{a}_1 \otimes \cdots \otimes \overline{a_i a_{i+1}} \otimes \cdots \otimes \bar{a}_{p+1}) + \\ & (-1)^{\epsilon_{p+1}} f(\bar{a}_1 \otimes \cdots \otimes \bar{a}_p) a_{p+1}, \end{aligned}$$

(2) for  $f \in \text{Hom}_{\mathbb{K}}((s^{-1}\bar{C})^{\otimes p}, A)^{n-k+1}$  and  $p > 0$ ,

$$\begin{aligned} \delta_1(f)(\bar{a}_1 \otimes \cdots \otimes \bar{a}_{p-1}) := & \mu(\text{id} \otimes f)(\Delta(1) \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_{p-1}) + \\ & \sum_{i=1}^{p-2} (-1)^{\epsilon_i} f(\bar{a}_1 \otimes \cdots \otimes \overline{\Delta(a_i)} \otimes \cdots \otimes \bar{a}_{p-1}) + \\ & (-1)^{\epsilon_n} \mu(f \otimes \text{id})(\bar{a}_1 \otimes \cdots \otimes \bar{a}_{p-1} \otimes \Delta(1)), \end{aligned}$$

(3) for  $f \in \text{Hom}_{\mathbb{K}}((s^{-1}\bar{C})^{\otimes p}, A)^{n-k+1}$  and  $p = 0$ ,

$$\delta_1(f) = \mu(\Delta(1) \otimes (f \otimes 1)).$$

We call  $(\mathcal{D}^*(A, A), \delta)$  the *Tate-Hochschild complex* of  $A$  and define the *Tate-Hochschild cohomology group* of degree  $n$ , denoted by  $\text{TH}^n(A, A)$ , for any  $n \in \mathbb{Z}$ , to be the cohomology group  $H^n(\mathcal{D}^*(A, A), \delta)$ . More explicitly,  $(\mathcal{D}^*(A, A), \delta)$  is the totalization complex of the double complex  $\mathcal{D}^{*,*}(A, A)$  given by

$$\cdots \rightarrow s^{1-k}C_{-1,*}(A, A) \rightarrow s^{1-k}C_{0,*}(A, A) \xrightarrow{\gamma} C^{0,*}(A, A) \rightarrow C^{1,*}(A, A) \rightarrow \cdots$$

where  $\gamma$  is the composition

$$s^{1-k}C_{0,*}(A, A) \cong s^{1-k}A \xrightarrow{\Delta} s(A \otimes A) \xrightarrow{T} s(A \otimes A) \xrightarrow{\mu} sA \xrightarrow{s^{-1}} A \cong C^{0,*}(A, A),$$

and by totalization we mean the direct sum totalization in the Hochschild chains direction and the direct product totalization in the Hochschild cochains direction.

**Proposition 3.11.** *Let  $A$  be a dg symmetric Frobenius algebra over a field  $\mathbb{K}$ . Then for any  $i \in \mathbb{Z}$ ,*

$$\text{TH}^i(A, A) \cong \text{HH}_{\text{sg}}^i(A, A).$$

*Proof.* From Remark 3.5 and Lemma 3.9, it follows that

$$\begin{aligned} \text{HH}_{\text{sg}}^i(A, A) &\cong \text{Hom}_{\mathcal{K}_{ac}}(A \otimes A^{\text{op}}\text{-Mod}_{\text{inj}})(S(A), s^i S(A)) \\ &\cong \text{Hom}_{\mathcal{K}_{ac}}(A \otimes A^{\text{op}}\text{-Mod}_{\text{inj}})(\text{cone}(\tau), s^i \text{cone}(\tau)) \\ &\cong \text{Hom}_{A \otimes A^{\text{op}}}(\text{cone}(\tau), s^i A) \\ &\cong H^i(\mathcal{D}^*(A, A)) \\ &= \text{TH}^i(A, A). \end{aligned}$$

□

**3.3. The singular Hochschild complex of a differential graded algebra.** In this subsection we fix a commutative ring  $\mathbb{K}$  and a differential graded  $\mathbb{K}$ -algebra  $A$  (not necessarily a dg symmetric Frobenius algebra).

Recall that we have defined a morphism of degree one of dg  $A$ - $A$ -bimodules (cf. (2)),

$$b_{-m} : \text{Bar}_{-m}(A) \rightarrow \text{Bar}_{-m+1}(A),$$

which induces a morphism of degree zero

$$sb_{-m} : s \text{Bar}_{-m}(A) \rightarrow \text{Bar}_{-m+1}(A).$$

Let us denote by  $\Omega^{m+1}(A)$  the kernel of  $sb_{-m}$ . In particular, we denote by  $\Omega^1(A)$  the kernel of  $s\mu : s(A \otimes A) \rightarrow sA$  where  $\mu$  is the multiplication of  $A$ , and we write  $\Omega^0(A) := A$ . Obviously,  $\Omega^m(A)$  is a dg  $A$ - $A$ -bimodule for any  $m \in \mathbb{Z}$ . Denote by  $b$  the differential of  $\text{Bar}_*(A)$  and by  $\pi : A \twoheadrightarrow s\bar{A}$  the natural projection map of degree one.

**Lemma 3.12.** *For each  $p \in \mathbb{Z}_{\geq 0}$ , there is an isomorphism of dg  $A$ - $A$ -bimodules*

$$\alpha : \Omega^p(A) \xrightarrow{\cong} (s\bar{A})^{\otimes p} \otimes A,$$

where the left  $A$ -module structure in  $(s\bar{A})^{\otimes p} \otimes A$  is given by

$$a \blacktriangleright (\bar{x}_1 \otimes \cdots \otimes \bar{x}_p \otimes x_{p+1}) := (\pi \otimes \text{id}^{\otimes p})(b(a \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_p \otimes x_{p+1})),$$

the right  $A$ -module structure is given by multiplication on the right  $A$  factor of  $(s\bar{A})^{\otimes p} \otimes A$ , and the differential on  $(s\bar{A})^{\otimes p} \otimes A$  is given by

$$d(\bar{x}_1 \otimes \cdots \otimes \bar{x}_p \otimes x_{p+1}) = \sum_{i=1}^{p+1} (-1)^{\epsilon_{i-1}} \bar{x}_1 \otimes \cdots \otimes d(\bar{x}_i) \otimes \cdots \otimes x_{p+1},$$

where  $\epsilon_{i-1} = \sum_{j=1}^{i-1} \deg(x_j) - i + 1$ .

*Proof.* It is easy to check that  $\blacktriangleright$  defines a dg left  $A$ -module structure on  $(s\bar{A})^{\otimes p} \otimes A$ , namely, for any  $a_1, a_2 \in A$ ,

$$a_1 \blacktriangleright (a_2 \blacktriangleright (\bar{x}_1 \otimes \cdots \otimes \bar{x}_p \otimes x_{p+1})) = (a_1 a_2) \blacktriangleright (\bar{x}_1 \otimes \cdots \otimes \bar{x}_p \otimes x_{p+1}).$$

The morphism  $\alpha$  is defined as the composition

$$\Omega^p(A) \hookrightarrow s(A \otimes (s\bar{A})^{\otimes p-1} \otimes A) \xrightarrow{\pi \otimes \text{id}^{\otimes p}} s\bar{A} \otimes (s\bar{A})^{\otimes p-1} \otimes A.$$

The inverse of  $\alpha$  is given by the morphism  $\beta$  defined by the composition

$$\beta : (s\bar{A})^{\otimes p} \otimes A \rightarrow A \otimes (s\bar{A})^{\otimes p} \otimes A \xrightarrow{b} \Omega^p(A),$$

where the first morphism is given by

$$\bar{x}_1 \otimes \cdots \otimes \bar{x}_p \otimes x_{p+1} \mapsto 1 \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_p \otimes x_{p+1}.$$

□

**Remark 3.13.** From now on, we identify  $\Omega^p(A)$  with  $(s\bar{A})^{\otimes p} \otimes A$  via the isomorphism  $\alpha$ .

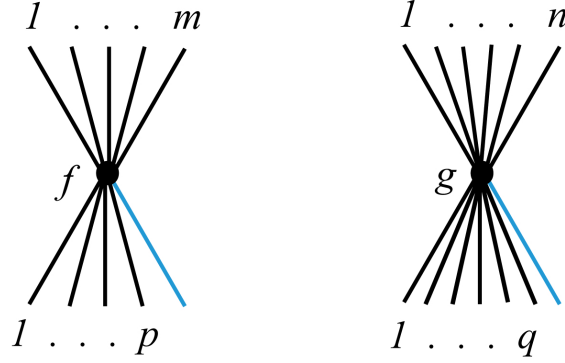


FIGURE 1. Using the identification of Lemma 3.12, we may represent diagrammatically elements  $f \in C^{m,*}(A, \Omega^p(A))$  and  $g \in C^{n,*}(A, \Omega^q(A))$  as corollas. For example  $f$  has  $m$  input legs at the top and  $p+1$  output legs at the bottom. The rightmost output is colored blue, indicating that such a leg represents an element of  $A$ , while black output legs should be interpreted as elements in  $s\bar{A}$ . Compositions of maps will be represented by stacking trees, as usual.

Consider the Hochschild cochain complex  $C^*(A, \Omega^p(A))$  with coefficients in the dg  $A$ - $A$ -bimodule  $\Omega^p(A)$ . Let us define a morphism

$$\tilde{\theta}_p : C^*(A, \Omega^p(A)) \rightarrow C^*(A, \Omega^{p+1}(A))$$

which sends an element  $f \in C^*(A, (s\bar{A})^{\otimes p} \otimes A)$  to  $\tilde{\theta}_{m,p}(f)$  given by the following formula,

$$(7) \quad \tilde{\theta}_p(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_{k+1}}) = (-1)^{\deg(x_1)\deg(f)} \overline{x_1} \otimes f(\overline{x_2} \otimes \cdots \otimes \overline{x_{k+1}}).$$

**Lemma 3.14.**  $\tilde{\theta}_p$  is a morphism of complexes (of degree zero) for each  $p \in \mathbb{Z}_{\geq 0}$ . Namely, the following diagram commutes

$$\begin{array}{ccc} C^*(A, \Omega^p(A)) & \xrightarrow{\tilde{\theta}_p} & C^*(A, \Omega^{p+1}(A)) \\ \downarrow \delta & & \downarrow \delta \\ C^*(A, \Omega^p(A)) & \xrightarrow{\tilde{\theta}_p} & C^*(A, \Omega^{p+1}(A)). \end{array}$$

*Proof.* For any  $f \in C^{m,*}(A, (s\bar{A})^{\otimes p} \otimes A)$  we have

$$\begin{aligned} \tilde{\theta}_p(\delta^h(f))(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}}) &= \pm \overline{x_1} \otimes \delta^h(f)(\overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}}) \\ &= \pm \overline{x_1} \otimes (\pi \otimes id)(b(\overline{x_2} \otimes f(\overline{x_3} \otimes \cdots \otimes \overline{x_{m+1}}))) \\ &\quad \pm \overline{x_1} \otimes f(b(\overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad \pm \overline{x_1} \otimes f(\overline{x_2} \otimes \cdots \otimes \overline{x_m}) \cdot x_m \end{aligned}$$

and

$$\begin{aligned} \delta^h(\tilde{\theta}_p(f))(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}}) &= \pm (\pi \otimes id)(b(\overline{x_1} \otimes \overline{x_2} \otimes f(\overline{x_3} \otimes \cdots \otimes \overline{x_{m+1}}))) \\ &\quad \pm \tilde{\theta}_p(f)(b(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad \pm \tilde{\theta}_p(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_m}) \cdot x_{m+1}. \end{aligned}$$

We may write the term  $\tilde{\theta}_p(f)(b(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}}))$  above as

$$\pm b(\overline{x_1} \otimes \overline{x_2}) \otimes f(\overline{x_3} \otimes \cdots \otimes \overline{x_{m+1}}) \pm \overline{x_1} \otimes f(b(\overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}})).$$

Thus

$$\begin{aligned} \delta^h(\tilde{\theta}_p(f))(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}}) &= \pm (\pi \otimes \text{id})(b(\overline{x_1} \otimes \overline{x_2} \otimes f(\overline{x_3} \otimes \cdots \otimes \overline{x_{m+1}}))) \\ &\quad \pm b(\overline{x_1} \otimes \overline{x_2}) \otimes f(\overline{x_3} \otimes \cdots \otimes \overline{x_{m+1}}) \\ &\quad \pm \overline{x_1} \otimes f(b(\overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad \pm \overline{x_1} \otimes f(\overline{x_2} \otimes \cdots \otimes \overline{x_m}) \cdot x_m. \end{aligned}$$

Since

$$\begin{aligned} \pm \overline{x_1} \otimes (\pi \otimes \text{id})(b(\overline{x_2} \otimes f(\overline{x_3} \otimes \cdots \otimes \overline{x_{m+1}}))) &= \pm (\pi \otimes \text{id})(b(\overline{x_1} \otimes \overline{x_2} \otimes f(\overline{x_3} \otimes \cdots \otimes \overline{x_{m+1}}))) \\ &\quad \pm b(\overline{x_1} \otimes \overline{x_2}) \otimes f(\overline{x_3} \otimes \cdots \otimes \overline{x_{m+1}}), \end{aligned}$$

it follows that  $\tilde{\theta}_p \circ \delta^h = \delta^h \circ \tilde{\theta}_p$ . A similar computation shows that  $\tilde{\theta}_p \circ \delta^v = \delta^v \circ \tilde{\theta}_p$ .  $\square$

It follows that we have an inductive system of complexes

$$\cdots \longrightarrow C^*(A, \Omega^p(A)) \xrightarrow{\tilde{\theta}_p} C^*(A, \Omega^{p+1}(A)) \xrightarrow{\tilde{\theta}_{p+1}} C^*(A, \Omega^{p+2}(A)) \longrightarrow \cdots$$

and we denote the colimit by

$$\mathcal{C}_{\text{sg}}^*(A, A) := \varinjlim_{p \in \mathbb{Z}_{\geq 0}} C^*(A, \Omega^p(A)).$$

Since the  $\tilde{\theta}_p$  are compatible with the differentials we obtain a differential

$$\delta_{\text{sg}}^m : \mathcal{C}_{\text{sg}}^m(A, A) \rightarrow \mathcal{C}_{\text{sg}}^{m+1}(A, A).$$

We call the complex  $(\mathcal{C}_{\text{sg}}^*(A, A), \delta_{\text{sg}})$  the *singular Hochschild cochain complex* of  $A$ .

By Lemma 2.10, for any  $p \in \mathbb{Z}_{\geq 0}$ , we have the following exact sequence of dg  $A$ - $A$ -bimodules

$$0 \rightarrow \Omega^{p+1}(A) \hookrightarrow s \text{Bar}_{-p}(A) \rightarrow s\Omega^p(A) \rightarrow 0.$$

Therefore, we may take the derived functor  $\text{HH}^*(A, -)$  in the derived category  $\mathcal{D}(A \otimes A^{\text{op}}\text{-Mod})$  to obtain a long exact sequence

$$\cdots \longrightarrow \text{HH}^m(A, s \text{Bar}_p(A)) \longrightarrow \text{HH}^m(A, s\Omega^p(A)) \xrightarrow{\theta_{m,p}} \text{HH}^{m+1}(A, \Omega^{p+1}(A)) \longrightarrow \cdots$$

where  $\theta_{m,p}$  denotes the connecting homomorphism. Since  $\text{HH}^m(A, s\Omega^p(A)) \cong \text{HH}^{m+1}(A, \Omega^p(A))$ , we get an inductive system for any  $m \in \mathbb{Z}$ ,

$$\cdots \longrightarrow \text{HH}^{m+1}(A, \Omega^p(A)) \xrightarrow{\theta_p} \text{HH}^{m+1}(A, \Omega^{p+1}(A)) \xrightarrow{\theta_{p+1}} \text{HH}^{m+1}(A, \Omega^{p+2}(A)) \longrightarrow \cdots$$

and denote its colimit by  $\varinjlim_{p \in \mathbb{Z}_{\geq 0}} \text{HH}^{m+1}(A, \Omega^p(A))$ .

**Lemma 3.15.** *For any  $p \in \mathbb{Z}_{\geq 0}$  and  $m \in \mathbb{Z}$ , we have that  $H^m(\tilde{\theta}_p) = \theta_p$ . Namely, the following diagram commutes*

$$\begin{array}{ccc} H^m(C^*(A, \Omega^p(A))) & \xrightarrow{H^m(\tilde{\theta}_p)} & H^m(C^*(A, \Omega^{p+1}(A))) \\ \downarrow \cong & & \downarrow \cong \\ \text{HH}^m(A, \Omega^p(A)) & \xrightarrow{\theta_p} & \text{HH}^m(A, \Omega^{p+1}(A)). \end{array}$$



*Proof.* First let us recall the construction of the connecting homomorphism

$$\theta_p : \mathrm{HH}^m(A, \Omega^p(A)) \rightarrow \mathrm{HH}^m(A, \Omega^{p+1}(A)).$$

Since the bar resolution  $B(A, A, A)$  is a projective resolution of  $A$ - $A$ -bimodule  $A$ , we have the following isomorphisms

$$\begin{aligned} \mathrm{HH}^m(A, \Omega^p(A)) &\cong \mathrm{Hom}_{\mathcal{D}(A \otimes A^{\mathrm{op}})}(A, s^m \Omega^p(A)) \\ &\cong \mathrm{Hom}_{\mathcal{K}(A \otimes A^{\mathrm{op}})}(B(A, A, A), s^m \Omega^p(A)). \end{aligned}$$

Any  $f \in \mathrm{Hom}_{\mathcal{K}(A \otimes A^{\mathrm{op}})}(B(A, A, A), s^m \Omega^p(A))$  may be lifted uniquely to  $\theta(f)$  so that the following diagram commutes

$$\begin{array}{ccc} B(A, A, A) & \xrightarrow{f} & s^m((s\bar{A})^{\otimes p} \otimes A) \\ \uparrow d & \searrow 1 \otimes \bar{f} & \uparrow \\ B(A, A, A) & & s^m \mathrm{Bar}_{-p}(A) \\ & \searrow \theta(f) & \uparrow \\ & & s^{m-1}((s\bar{A})^{\otimes p+1} \otimes A) \end{array}$$

where  $d : B(A, A, A) \rightarrow B(A, A, A)$  is the differential of the two sided bar construction of  $A$ , the vertical maps in the right column are the two middle maps in the short exact sequence  $0 \rightarrow s^{m-1} \Omega^{p+1}(A) \hookrightarrow s^m \mathrm{Bar}_{-p}(A) \rightarrow s^m \Omega^p(A) \rightarrow 0$ , and  $\bar{f} : T(s\bar{A}) \otimes A \rightarrow s^m \Omega^p(A)$  is defined by

$$\bar{f}(\overline{a_1} \otimes \dots \otimes \overline{a_{m+p+1}} \otimes a_{m+p+1}) = f(1 \otimes \overline{a_1} \otimes \dots \otimes \overline{a_{m+1}} \otimes a_{m+p+1}).$$

The map  $\theta$  is precisely the connection homomorphism in the long exact sequence. But note if we place the map  $H(\tilde{\theta})$  in the dotted morphism it also makes the diagram commute, so by uniqueness it follows that  $H(\tilde{\theta}) = \theta$ .  $\square$

**Proposition 3.16.** *For any  $m \in \mathbb{Z}$ , we have a natural isomorphism*

$$H^m(\mathcal{C}_{\mathrm{sg}}^*(A, A)) \cong \varinjlim_{p \in \mathbb{Z}_{\geq 0}} \mathrm{HH}^m(A, \Omega^p(A)).$$

*In particular, via such an isomorphism, the quotient functor  $\mathcal{D}^b(A \otimes A^{\mathrm{op}}) \rightarrow \mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$  induces a natural morphism*

$$\chi : H^*(\mathcal{C}_{\mathrm{sg}}^*(A, A)) \rightarrow \mathrm{HH}_{\mathrm{sg}}^*(A, A).$$

*Proof.* The first isomorphism follows from Lemma 3.15 and the fact that the homology functor commutes with colimit. For all  $p \in \mathbb{Z}_{\geq 0}$  we have a natural morphism

$$\mathrm{HH}^*(A, \Omega^p(A)) \rightarrow \mathrm{HH}_{\mathrm{sg}}^*(A, A)$$

induced from the quotient functor  $\mathcal{D}^b(A \otimes A^{\mathrm{op}}) \rightarrow \mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$  and the isomorphism  $\Omega^p(A) \cong A$  in  $\mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$  (since  $\Omega^1(A) \cong A$  in  $\mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$ ). These morphisms are compatible with the structure maps  $\theta_{m,p}$  and thus induce a natural morphism  $\chi : H^*(\mathcal{C}_{\mathrm{sg}}^*(A, A)) \rightarrow \mathrm{HH}_{\mathrm{sg}}^*(A, A)$ .  $\square$

**3.4. A homotopy retract between  $\mathcal{D}^*(A, A)$  and  $\mathcal{C}_{\text{sg}}^*(A, A)$ .** In this subsection we assume that  $(A, d, \mu, \Delta)$  is a dg symmetric Frobenius algebra of degree  $k$  ( $k > 0$ ). We will construct a (strong) homotopy retract (cf. [LoVa, Section 10]) of complexes,

$$\begin{aligned} \pi_{m,p}(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}}) &:= \sum_i (-1)^{\deg(e_i)(\deg(f)+1-k)} e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}})) \\ &= \sum_i (-1)^{\deg(e_i)(\deg(f)+1-k)} (\pi \otimes \text{id}^{\otimes p-1})b(e_i \otimes (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}}))), \end{aligned}$$

where we recall that  $\epsilon : A \rightarrow \mathbb{K}$  is the counit and the  $\blacktriangleright$ -action is defined in Lemma 3.12.

Define  $\pi_{m,0} = \text{id} : C^{m,*}(A, A) \rightarrow C^{m,*}(A, A)$  for  $m > 0$  and for  $m = 0, p \in \mathbb{Z}_{>0}$  define

$$\pi_{0,p} : C^{0,*}(A, \Omega^p(A)) \rightarrow C^{-(p-1),*}(A, A)$$

as follows: for any  $x := \overline{x_1} \otimes \cdots \otimes \overline{x_p} \otimes x_{p+1} \in (s\overline{A})^{\otimes p} \otimes A$  let

$$\pi_{0,p}(x) := (\epsilon(\overline{x_1})\overline{x_2} \otimes \cdots \otimes \overline{x_p} \otimes x_{p+1}).$$

**Remark 3.18.** Since  $\epsilon(1) = 0$ , it follows that the counit induces a well-defined map  $\epsilon : \overline{A} \rightarrow k$ . In Figures 2 and 3 below we give a diagrammatic representation of the maps

$$\iota : C_{-m,*}(A, A) \rightarrow C^0(A, (s\overline{A})^{\otimes m+1} \otimes A)$$

and

$$\pi_{m,p} : C^{m,*}(A, (s\overline{A})^{\otimes p} \otimes A) \rightarrow C^{m-1,*}(A, (s\overline{A})^{\otimes p-1} \otimes A),$$

the latter in the case  $m, p \in \mathbb{Z}_{>0}$ . We represent an element of  $C_{-m,*}(A, A)$  as a corolla with no inputs (this is what the solid black circle in the top leg means) and  $m+1$  output legs  $m$  which are colored black indicating these legs represent an element in  $(s\overline{A})^{\otimes m}$  and one blue leg which represents an element of  $A$ . A blue circle with white interior represents the unit 1 of the algebra  $A$ . We denote by  $\Delta : A \rightarrow A \otimes A$  the coproduct of the dg symmetric Frobenius algebra  $A$ , by  $\pi : A \rightarrow s\overline{A}$  the natural projection map, by  $\epsilon : s\overline{A} \rightarrow \mathbb{K}$  the map induced by the counit, and by  $\mu$  the product of  $A$ .

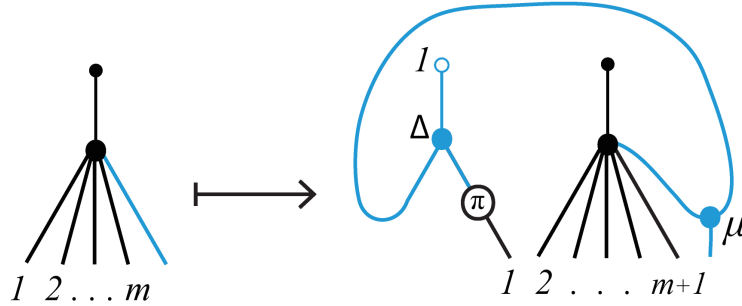


FIGURE 2.  $\iota : C_{-m,*}(A, A) \rightarrow C^0(A, (s\overline{A})^{\otimes m+1} \otimes A) \cong \text{Hom}_{\mathbb{K}}(\mathbb{K}, (s\overline{A})^{\otimes m+1} \otimes A)$

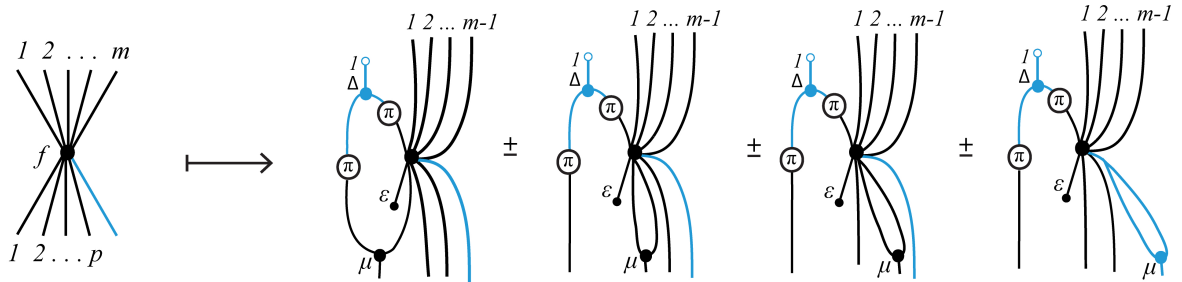


FIGURE 3.  $\pi_{m,p} : C^{m,*}(A, (s\overline{A})^{\otimes p} \otimes A) \rightarrow C^{m-1,*}(A, (s\overline{A})^{\otimes p-1} \otimes A)$

**Lemma 3.19.**  $\pi_{>0,*}$  is compatible with the differentials.

*Proof.* First of all, let us check that  $\pi_{>0,*}$  is compatible with the external differentials. Let  $f \in C^{m+1,*}(A, \Omega^p(A))$ ,  $(m > -1)$ . Then we have for any  $\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}} \in (s\overline{A})^{\otimes m+1}$

$$\begin{aligned} \pi_{*,*} \circ \delta_1(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}}) &= \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(\delta_1(f)(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &= \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad + \sum_{j=1}^m \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_j x_{j+1}} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad + \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f_i \blacktriangleright f(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad + \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_m}) x_{m+1}) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \delta_1 \circ \pi_{*,*}(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}}) &= \sum_{j=1}^m \pm \pi_{*,*}(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_j x_{1+j}} \otimes \cdots \otimes \overline{x_{m+1}}) \\ &\quad \pm x_1 \blacktriangleright (\pi_{*,*}(f)(\overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}})) \pm (\pi_{*,*}(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_m})) x_{m+1} \\ &= \sum_{j=1}^m \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_j x_{j+1}} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad + x_1 \blacktriangleright (\sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}}))) \\ &\quad + (\sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_m}))) x_{m+1}. \end{aligned}$$

Thus, we may cancel terms to obtain:

$$\begin{aligned} (\pi_{*,*} \circ \delta_1 - \delta_1 \circ \pi_{*,*})(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}}) &= \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad + \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f_i \blacktriangleright f(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}})) \\ &\quad + x_1 \blacktriangleright (\sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}}))). \end{aligned}$$

From the fact that  $\blacktriangleright$  defines a left action of  $A$  on  $\Omega^p(A)$ , it follows that

$$x_1 \blacktriangleright (\sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}}))) = \sum_i \pm x_1 e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_{m+1}})),$$

thus the first sums cancel with the last sum since  $\sum x_1 e_i \otimes f_i = \sum e_i \otimes f_i x_1$ . Also, it follows from  $\sum_i (-1)^{\deg(e_i)k} \overline{e_i} \otimes \epsilon(\overline{f_i}) = \sum_i (-1)^{\deg(e_i)k} \overline{e_i} \epsilon(\overline{f_i}) = 0$  that the second sum vanishes, namely, we have

$$\sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f_i \blacktriangleright f(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}})) = 0.$$

Therefore,  $(\pi_{*,*} \circ \delta_1 - \delta_1 \circ \pi_{*,*})(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_{m+1}}) = 0$ . Similarly, we may check that  $\pi_{>0,*}$  is compatible with the internal differentials. Let  $f \in C^{m+1,*}(A, \Omega^p(A))$  for  $m > -1$ ,

then

$$\begin{aligned} \pi_{*,*} \circ \delta_0(f)(\overline{x}_1 \otimes \cdots \otimes \overline{x}_m) &= \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(df(\overline{f}_i \otimes \overline{x}_1 \otimes \cdots \otimes \overline{x}_m)) \\ &\quad + \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(d(\overline{f}_i) \otimes \overline{x}_1 \otimes \cdots \otimes \overline{x}_m)) \\ &\quad + \sum_{j=1}^m \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(f_i \otimes \overline{x}_1 \otimes \cdots \otimes d(x_i) \otimes \cdots \otimes \overline{x}_m)) \end{aligned}$$

and

$$\begin{aligned} \delta_0 \circ \pi_{*,*}(f)(\overline{x}_1 \otimes \cdots \otimes \overline{x}_m) &= d\left(\sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f}_i \otimes \overline{x}_1 \otimes \cdots \otimes \overline{x}_m))\right) \\ &\quad + \sum_{j=1}^m \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f}_i \otimes \overline{x}_1 \otimes \cdots \otimes d(\overline{x}_i) \otimes \cdots \otimes \overline{x}_m)), \end{aligned}$$

thus we may cancel terms to obtain

$$\begin{aligned} (\pi_{*,*} \circ \delta_0(f) - \delta_0 \circ \pi_{*,*})(\overline{x}_1 \otimes \cdots \otimes \overline{x}_m) &= \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(df(\overline{f}_i \otimes \overline{x}_1 \otimes \cdots \otimes \overline{x}_m)) \\ &\quad + \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(d(\overline{f}_i) \otimes \overline{x}_1 \otimes \cdots \otimes \overline{x}_m)) \\ &\quad + d\left(\sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f}_i \otimes \overline{x}_1 \otimes \cdots \otimes \overline{x}_m))\right). \end{aligned}$$

Using  $d(\sum_i e_i \otimes f_i) = 0$  we may conclude that the three sums in the above expression vanish. Hence the maps  $\pi_{>0,*}$  are compatible with the internal differentials and thus compatible with the differentials.  $\square$

**Remark 3.20.** By a similar computation, we have that  $\pi_{0,*}$  are compatible with the internal differentials. Take an element

$$x := \overline{x}_1 \otimes \cdots \otimes \overline{x}_p \otimes x_{p+1} \in (s\overline{A})^{\otimes p} \otimes A,$$

then

$$\begin{aligned} \pi_{0,p} \circ d(x) &= \pi_{0,p}(d(\overline{x}_1) \otimes \overline{x}_2 \otimes \cdots \otimes \overline{x}_p \otimes x_{p+1}) \\ &\quad + \pi_{0,p}\left(\sum_{i=2}^{p+1} \pm \overline{x}_1 \otimes \cdots \otimes d(\overline{x}_i) \otimes \cdots \otimes x_{p+1}\right) \\ &= \epsilon(d(\overline{x}_1))\overline{x}_2 \otimes \cdots \otimes \overline{x}_p \otimes x_{p+1} \\ &\quad + \sum_{i=2}^{p+1} \pm \epsilon(\overline{x}_1)\overline{x}_2 \otimes \cdots \otimes d(\overline{x}_i) \otimes \cdots \otimes x_{p+1} \\ &= \epsilon(\overline{x}_1)d(\overline{x}_2) \otimes \overline{x}_3 \otimes \cdots \otimes \overline{x}_p \otimes x_{p+1} \\ &\quad + \sum_{i=3}^{p+1} \pm \epsilon(\overline{x}_1)\overline{x}_2 \otimes \overline{x}_3 \otimes \cdots \otimes d(\overline{x}_i) \otimes \cdots \otimes x_{p+1} \\ &= d \circ \pi_{0,p}(x). \end{aligned}$$

Let us check that  $\pi_{0,*}$  are also compatible with the external differentials. Namely, that the following diagram commutes for any  $p \in \mathbb{Z}_{>0}$ :

$$(9) \quad \begin{array}{ccc} C^{0,*}(A, \Omega^p(A)) & \xrightarrow{\pi_{0,p}} & C_{-(p-1),*}(A, A) \\ \downarrow \delta_1 & & \searrow \delta_1 \\ C^{1,*}(A, \Omega^p(A)) & \xrightarrow{\pi_{1,p}} & C^{0,*}(A, \Omega^{p-1}(A)) \xrightarrow{\pi_{0,p-1}} C_{-(p-2),*}(A, A). \end{array}$$

The commutativity of diagram (9) follows since

$$\begin{aligned} \pi_{0,p-1} \circ \pi_{1,p} \circ \delta_1(\overline{x_1} \otimes \cdots \otimes \overline{x_p} \otimes x_{p+1}) &= \sum_i \pi_{0,p-1}((\pi \otimes \text{id})(b(e_i \epsilon(\overline{x_1}) \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_p} \otimes x_{p+1} f_i))) \\ &= \sum_i \epsilon(\overline{e_i x_2}) \epsilon(\overline{x_1}) \otimes \overline{x_3} \otimes \cdots \otimes \overline{x_p} \otimes x_{p+1} f_i \\ &\quad \pm \sum_i \sum_{j=1}^{p-1} \epsilon(\overline{e_i}) \epsilon(\overline{x_1}) \overline{x_2} \otimes \cdots \otimes \overline{x_{j+1} x_{j+2}} \otimes \cdots \otimes x_{p+1} f_i \\ &= \sum_i \epsilon(\overline{x_1}) \overline{x_3} \otimes \cdots \otimes \overline{x_p} \otimes x_{p+1} x_2 \\ &\quad + \sum_i \sum_{j=1}^{p-1} \epsilon(\overline{x_1}) \overline{x_2} \otimes \cdots \otimes \overline{x_{j+1} x_{j+2}} \otimes \cdots \otimes x_{p+1} f_i \\ &= \delta_1 \circ \pi_{0,p}(\overline{x_1} \otimes \cdots \otimes \overline{x_p} \otimes x_{p+1}), \end{aligned}$$

where the third identity follows from the facts  $\sum_i \epsilon(\overline{e_i}) f_i = 1$  and  $\sum_i x e_i \otimes f_i = \sum_i e_i \otimes f_i x$ .

**Lemma 3.21.** *For  $m, p \in \mathbb{Z}_{>0}$  we have*

$$\pi_{m,p} \circ \theta_{m-1,p-1} = \text{id}.$$

For  $m = 0, p \in \mathbb{Z}_{\geq 0}$ , we have

$$\pi_{0,p} \circ \iota = \text{id}.$$

*Proof.* Recall that for any  $f \in C^{m-1,*}(A, (s\overline{A})^{\otimes p-1} \otimes A)$ ,

$$\theta_{m-1,p-1} : C^{m-1,*}(A, \Omega^{p-1}(A)) \rightarrow C^{m,*}(A, \Omega^p(A))$$

is defined by

$$\theta_{m-1,p-1}(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_m}) := (-1)^{\deg(x_1)\deg(f)} \overline{x_1} \otimes f(\overline{x_2} \otimes \cdots \otimes \overline{x_m}).$$

Thus

$$\begin{aligned} \pi_{m,p} \circ \theta_{m-1,p-1}(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}}) &= \sum_i \pm e_i \blacktriangleright (\epsilon \otimes \text{id}^{\otimes p})(\theta_{m-1,p-1}(f)(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}})) \\ &= \sum_i \pm e_i \blacktriangleright (\epsilon(\overline{f_i}) f(\overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}})) \\ &= f(\overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}}). \end{aligned}$$

Similarly, let  $\overline{x_1} \otimes \cdots \otimes \overline{x_{p-1}} \otimes x_p \in C_{-(p-1),*}(A, A)$ , then we have

$$\begin{aligned} \pi_{0,p} \circ \iota(\overline{x_1} \otimes \cdots \otimes \overline{x_{p-1}} \otimes x_p) &= \sum_i \pm \pi_{0,p}(\overline{e_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_{p-1}} \otimes x_p f_i) \\ &= \sum_i \pm \epsilon(\overline{e_i}) \overline{x_1} \otimes \cdots \otimes \overline{x_{p-1}} \otimes x_p f_i \\ &= \overline{x_1} \otimes \cdots \otimes \overline{x_{p-1}} \otimes x_p. \end{aligned}$$

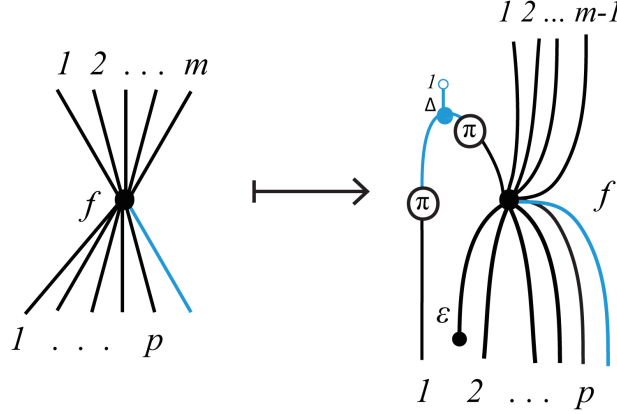


FIGURE 4. The above diagram represents the map  $h_{m,p} : C^{m,*}(A, \Omega^p(A)) \rightarrow C^{m-1,*}(A, \Omega^p(A))$ .

□

**Definition 3.22.** We define  $\Pi : \mathcal{C}_{\text{sg}}^*(A, A) \rightarrow \mathcal{D}^*(A, A)$  as follows: For an element  $\bar{f} \in \mathcal{C}_{\text{sg}}^*(A, A)$  represented by an element  $f \in C^{m,*}(A, \Omega^p(A))$ , let

$$\Pi(\bar{f}) := \begin{cases} \pi_{m-p,0} \circ \pi_{m-p+1,1} \circ \cdots \circ \pi_{m,p}(f) & \text{if } m-p \geq 0, \\ \pi_{0,p-m} \circ \pi_{1,m-p+1} \circ \cdots \circ \pi_{m,p}(f) & \text{if } m-p < 0. \end{cases}$$

From Lemma 3.21, this is indeed well-defined, namely,  $\Pi$  does not depend on the representative for  $\bar{f}$ . Moreover, it follows from Lemma 3.19 and Lemma 3.21 that  $\Pi$  is a morphism of (degree zero) chain complexes such that  $\Pi \circ \iota = \text{id}$ .

Finally, we construct the chain homotopy  $h : \mathcal{C}_{\text{sg}}^*(A, A) \rightarrow s^{-1}\mathcal{C}_{\text{sg}}^*(A, A)$ . For  $m \in \mathbb{Z}_{>0}, p \in \mathbb{Z}_{>0}$ , we define a linear map

$$h_{m,p} : C^{m,*}(A, \Omega^p(A)) \rightarrow C^{m-1,*}(A, \Omega^p(A))$$

which sends  $f \in C^{m,*}(A, (s\bar{A})^{\otimes p} \otimes A)$  to

$$\bar{x}_1 \otimes \cdots \otimes \bar{x}_{m-1} \mapsto \sum_i (-1)^{\deg(e_i)(\deg(f)+1-k)} \bar{e}_i \otimes (\epsilon \otimes \text{id}^{\otimes p})(f(\bar{f}_i \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_{m-1})).$$

We also define  $h_{m,p} := 0$  for  $mp = 0$ . Note that the total degree of the map  $h_{m,p}$  is  $-1$ . For  $m \in \mathbb{Z}_{>0}$  and  $p \in \mathbb{Z}_{\geq 0}$  define

$$H_{m,p} : C^{m,*}(A, \Omega^p(A)) \rightarrow C^{m-1,*}(A, \Omega^p(A))$$

as the composition

$$H_{m,p} := \sum_{i=0}^{\min\{p,m\}} \theta_{m-2,p-1} \circ \cdots \circ \theta_{m-i-1,p-i} \circ h_{m-i,p-i} \circ \pi_{m-i+1,p-i+1} \circ \cdots \circ \pi_{m,p}.$$

Otherwise,  $H_{m,p} := 0$ . See Figure 4.

**Lemma-Definition 3.23.** The following identity holds for  $m, p \in \mathbb{Z}_{>0}$

$$H_{m,p} \circ \theta_{m-1,p-1} = \theta_{m-2,p-1} \circ H_{m-1,p-1}.$$

Namely the following diagram commutes

$$\begin{array}{ccc} C^{m-1,*}(A, \Omega^{p-1}(A)) & \xrightarrow{\theta_{m-1,p-1}} & C^{m,*}(A, \Omega^p(A)) \\ \downarrow H_{m-1,p-1} & & \downarrow H_{m,p} \\ C^{m-2,*}(A, \Omega^{p-1}(A)) & \xrightarrow{\theta_{m-2,p-1}} & C^{m-1,*}(A, \Omega^p(A)). \end{array}$$

As a consequence,  $H_{*,*}$  induces a well-defined morphism

$$h : C_{\text{sg}}^*(A, A) \rightarrow s^{-1}C_{\text{sg}}^*(A, A).$$

*Proof.* We have that

$$\begin{aligned} H_{m,p} \circ \theta_{m-1,p-1} &= \sum_{i=1}^{\min\{p,m\}} \theta_{m-2,p-1} \circ \cdots \circ \theta_{m-i-1,p-i} \circ h_{m-i,p-i} \circ \pi_{m-i+1,p-i+1} \circ \cdots \circ \pi_{m,p} \circ \theta_{m-1,p-1} \\ &= \sum_{i=1}^{\min\{p,m\}} \theta_{m-2,p-1} \circ \cdots \circ \theta_{m-i-1,p-i} \circ h_{m-i,p-i} \circ \pi_{m-i+1,p-i+1} \circ \cdots \circ \pi_{m-1,p-1} \\ &= \theta_{m-2,p-1} \circ H_{m-1,p-1} \end{aligned}$$

where the second identity follows from Lemma 3.21.  $\square$

**Proposition 3.24.** *The map  $h$  is a chain homotopy between  $\text{id}$  and  $\iota \circ \Pi$ . Namely,*

$$\text{id} - \iota \circ \Pi = \delta \circ h + h \circ \delta.$$

*Proof.* Let us first prove the following identity for  $m \in \mathbb{Z}_{\geq 0}$  and  $p \in \mathbb{Z}_{> 0}$

$$(10) \quad \text{id} - \theta_{m-1,p-1} \circ \pi_{m,p} = \delta \circ h_{m,p} + h_{m+1,p} \circ \delta.$$

We observe that  $h_{*,*}$  are compatible with the internal differentials (but not with the external differentials), so it is sufficient to prove that we have the following homotopy diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^{0,*}(A, \Omega^p(A)) & \longrightarrow & C^{1,*}(A, \Omega^p(A)) & \longrightarrow & C^{2,*}(A, \Omega^p(A)) \longrightarrow \cdots \\ & & \downarrow \text{id} - \iota\pi & \swarrow h_{1,p} & \downarrow \text{id} - \theta\pi & \swarrow h_{2,p} & \downarrow \text{id} - \theta\pi \\ 0 & \longrightarrow & C^{0,*}(A, \Omega^p(A)) & \longrightarrow & C^{1,*}(A, \Omega^p(A)) & \longrightarrow & C^{2,*}(A, \Omega^p(A)) \longrightarrow \cdots \end{array}$$

Take an element  $x := \overline{x}_1 \otimes \cdots \otimes \overline{x}_p \otimes x_{p+1} \in C^{0,*}(A, (s\overline{A})^{\otimes p} \otimes A)$ , then

$$(\text{id} - \iota \circ \Pi)(x) = x - \sum_i \pm \overline{e}_i \otimes \epsilon(\overline{x}_1)\overline{x}_2 \otimes \cdots \otimes \overline{x}_p \otimes x_{p+1} f_i$$

and

$$\begin{aligned} h_{1,p} \circ \delta(x) &= \sum_i \pm \overline{e}_i \otimes (\epsilon \otimes \text{id}^{\otimes p})(\delta(x)(\overline{f}_i)) \\ &= \sum_i \pm \overline{e}_i \otimes (\epsilon \otimes \text{id}^{\otimes p})(\overline{f}_i \blacktriangleright (x)) + \overline{e}_i \otimes (\epsilon \otimes \text{id}^{\otimes p})x \overline{f}_i \\ &= x - \sum_i \pm \overline{e}_i \otimes \epsilon(\overline{x}_1)\overline{x}_2 \otimes \cdots \otimes \overline{x}_p \otimes x_{p+1} f_i \\ &= \text{id} - \iota \circ \Pi. \end{aligned}$$

Similarly, for  $m > 0$  we have

$$\begin{aligned} &(\text{id} - \theta_{m-1,p-1} \circ \pi_{m,p})(f)(\overline{x}_1 \otimes \cdots \otimes \overline{x}_m) \\ &= f(\overline{x}_1 \otimes \cdots \otimes \overline{x}_m) - \sum_i \pm \overline{x}_1 \otimes e_i \blacktriangleright (\epsilon \otimes \text{id})(f(\overline{f}_i \otimes \overline{x}_2 \otimes \cdots \otimes \overline{x}_m)). \end{aligned}$$



On the other hand, we have

$$\begin{aligned}
& (\delta \circ h_{m,p} + h_{m+1,p} \circ \delta)(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_m}) \\
&= \sum_i \pm \overline{x_1} \blacktriangleright (\overline{e_i} \otimes (\epsilon \otimes \text{id})(f(\overline{f_i} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_m}))) \\
&+ \sum_j \sum_{i=1}^{m-1} \pm \overline{e_j} \otimes (\epsilon \otimes \text{id})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_i x_{i+1}} \otimes \cdots \otimes \overline{x_m})) \\
&+ \sum_i \pm \overline{e_i} \otimes (\epsilon \otimes \text{id})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}}) \overline{x_m}) \\
&+ \sum_i \pm \overline{e_i} \otimes (\epsilon \otimes \text{id}^{\otimes p})(\overline{f_i} \blacktriangleright f(\overline{x_1} \otimes \cdots \otimes \overline{x_m})) \\
&+ \sum_i \pm \overline{e_i} \otimes (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i x_1} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_m})) \\
&+ \sum_{j=1}^{m-1} \sum_i \pm \overline{e_i} \otimes (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_j x_{j+1}} \otimes \cdots \otimes \overline{x_m})) \\
&+ \sum_i \pm \overline{e_i} \otimes (\epsilon \otimes \text{id})(f(\overline{f_i} \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_{m-1}}) \overline{x_m}) \\
&= \sum_i \pm \overline{x_1} \blacktriangleright (\overline{e_i} \otimes (\epsilon \otimes \text{id})(f(\overline{f_i} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_m}))) \\
&+ \sum_i \pm \overline{e_i} \otimes (\epsilon \otimes \text{id}^{\otimes p})(\overline{f_i} \blacktriangleright f(\overline{x_1} \otimes \cdots \otimes \overline{x_m})) \\
&+ \sum_i \pm \overline{e_i} \otimes (\epsilon \otimes \text{id}^{\otimes p})(f(\overline{f_i x_1} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_m})) \\
&= f(\overline{x_1} \otimes \cdots \otimes \overline{x_m}) - \sum_i \pm \overline{x_1} \otimes e_i \blacktriangleright (\epsilon \otimes \text{id})(f(\overline{f_i} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_m})) \\
&= (\text{id} - \theta_{m-1,p-1} \circ \pi_{m,p})(f)(\overline{x_1} \otimes \cdots \otimes \overline{x_m})
\end{aligned}$$

verifying identity (10). By induction we may conclude that

$$\text{id} - \iota \circ \Pi = \delta \circ h + h \circ \delta.$$

□

**Corollary 3.25.** *Let  $A$  be a dg symmetric Frobenius algebra over a field  $\mathbb{K}$ . Then the morphism (cf. Proposition 3.16)*

$$\chi : H^*(\mathcal{C}_{\text{sg}}^*(A, A)) \rightarrow \text{HH}_{\text{sg}}^*(A, A)$$

*is an isomorphism.*

*Proof.* We claim that the following diagram commutes

$$(11) \quad \begin{array}{ccc}
H^*(\mathcal{C}_{\text{sg}}^*(A, A)) & \xrightarrow{\chi} & \text{HH}_{\text{sg}}^*(A, A) \\
\swarrow \cong & & \searrow \cong \\
& H^*(\mathcal{D}^*(A, A)) & \\
\uparrow H^*(\iota) & & \uparrow \chi'
\end{array}$$

where the isomorphism  $\chi' : H^*(\mathcal{D}^*(A, A)) \rightarrow \text{HH}_{\text{sg}}^*(A, A)$  is given in Proposition 3.11. Indeed, recall that for any  $m \in \mathbb{Z}$ ,  $\chi'$  can be written as the composition of the following

morphisms

$$H^m(\mathcal{D}^*(A, A)) \rightarrow \text{Hom}_{\mathcal{K}_{ac}}(\text{cone}(\tau), s^m \text{cone}(\tau)) \xrightarrow{S^{-1}} \text{Hom}_{\mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})}(A, s^m A)$$

where for simplicity  $\mathcal{K}_{ac}$  denotes  $\mathcal{K}_{ac}(A \otimes A^{\text{op}}\text{-Mod}_{\text{inj}})$ . If  $\alpha \in H^m(\mathcal{D}^*(A, A))$  is any homogeneous element, then  $H^*(\iota)(\alpha)$  is represented by an element  $\alpha' \in \text{HH}^{m+p}(A, \Omega^p(A))$  for large enough  $p > 0$ . Denote by  $\beta$  the image of  $\alpha'$  via the following composition

$$\text{HH}^{m+p}(A, \Omega^p(A)) \rightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})}(A, s^{m+p} \Omega^p(A)) \xrightarrow{S} \text{Hom}_{\mathcal{K}_{ac}}(S(A), s^{m+p} S(\Omega^p(A))).$$

Note that for any  $i \in \mathbb{Z}$ ,  $S^i(\Omega^p(A)) \cong s^{i-p} S(A) \cong s^{i-p} \text{cone}(\tau)$  in  $\mathcal{K}_{ac}(A \otimes A^{\text{op}}\text{-Mod}_{\text{inj}})$ , hence we have the following isomorphism

$$\text{Hom}_{\mathcal{K}_{ac}}(S(A), s^{m+p} S(\Omega^p(A))) \cong \text{Hom}_{\mathcal{K}_{ac}}(\text{cone}(\tau), s^m \text{cone}(\tau)).$$

Let  $\beta' \in \text{Hom}_{\mathcal{K}_{ac}}(\text{cone}(\tau), s^m \text{cone}(\tau))$  denote the image of  $\beta$  via the isomorphism above. It is easy to check that  $\beta'$  is also the image of  $\alpha$  via the isomorphism  $S \circ \chi'$ , so the triangle diagram commutes. On the other hand, from the chain homotopy retraction described above it follows that  $H^*(\iota)$  is an isomorphism, thus  $\chi$  is an isomorphism as well.  $\square$

#### 4. DGA AND DGLA STRUCTURES ON THE SINGULAR HOCHSCHILD COMPLEX $\mathcal{C}_{\text{sg}}^*(A, A)$

In this section we recall briefly natural dga and dglA structures on the singular Hochschild complex  $\mathcal{C}_{\text{sg}}^*(A, A)$  for a differential graded algebra  $A$  over a commutative ring  $\mathbb{K}$ . All the constructions for dg algebras in this section are the dg generalization of the ones for (ordinary) associative algebras in [Wan1]. For more details, we refer to [Wan1]. Throughout this section, we assume that  $A$  is a differential graded associative algebra (not necessary symmetric Frobenius) over a commutative ring  $\mathbb{K}$ .

**4.1. DGA structure on  $\mathcal{C}_{\text{sg}}^*(A, A)$ .** Let  $f \in C^{m,*}(A, \Omega^p(A))$  and  $g \in C^{n,*}(A, \Omega^q(A))$ , we define the cup product  $f \cup g \in C^{m+n,*}(A, \Omega^{p+q}(A))$  by

$$f \cup g := (\text{id}^{\otimes p+q} \otimes \mu) \circ (\text{id}^{\otimes q} \otimes f \otimes \text{id}) \circ (\text{id}^{\otimes m} \otimes g);$$

where we identify  $\Omega^p(A)$  with  $s\bar{A}^{\otimes p} \otimes A$  as in Lemma 3.12 and denote by  $\mu : A \otimes A \rightarrow A$  the multiplication in the algebra  $A$ . In particular, when  $p = q = 0$  the cup product coincides with the usual one on Hochschild cochain complex  $\mathcal{C}^*(A, A)$ . The cup product  $\cup$  is clearly compatible with the structure maps  $\tilde{\theta}_p : C^*(A, \Omega^p(A)) \rightarrow C^*(A, \Omega^{p+1}(A))$  defined in the previous section, thus it induces a well-defined product

$$\cup : \mathcal{C}_{\text{sg}}^*(A, A) \otimes \mathcal{C}_{\text{sg}}^*(A, A) \rightarrow \mathcal{C}_{\text{sg}}^*(A, A).$$

**Proposition 4.1.** *The cup product  $\cup$  defined above gives a (unital) dg algebra structure on the singular Hochschild complex  $\mathcal{C}_{\text{sg}}^*(A, A)$ . Moreover, this cup product  $\cup$  is compatible with the Yoneda product of  $\text{HH}_{\text{sg}}^*(A, A)$  via the canonical morphism  $\chi : H^*(\mathcal{C}_{\text{sg}}^*(A, A)) \rightarrow \text{HH}_{\text{sg}}^*(A, A)$  (cf. Proposition 3.16).*

*Proof.* It is straightforward to verify that the cup product is associative and compatible with the differential. Note that since  $\Omega^p(A)$  is an  $A$ - $A$ -bimodule for all  $p \in \mathbb{Z}_{\geq 0}$  we have the Yoneda product

$$\cup' : \text{HH}^m(A, \Omega^p(A)) \otimes \text{HH}^n(A, \Omega^q(A)) \rightarrow \text{HH}^{m+n}(A, \Omega^p(A) \otimes_A \Omega^q(A)) \cong \text{HH}^{m+n}(A, \Omega^{p+q}(A))$$

defined through the classical Hochschild cup product construction. More precisely, take elements  $\bar{f} \in \text{HH}^m(A, \Omega^p(A))$  and  $\bar{g} \in \text{HH}^n(A, \Omega^q(A))$ , which are represented by  $f \in C^{m,*}(A, \Omega^p(A))$  and  $g \in C^{n,*}(A, \Omega^q(A))$  respectively, then  $\bar{f} \cup' \bar{g}$  is represented by

$$f \cup' g(\bar{a}_1 \otimes \cdots \otimes \bar{a}_{m'+n'}) := f(\bar{a}_1 \otimes \cdots \otimes \bar{a}_{m'}) \otimes_A g(\bar{a}_{m'+1} \otimes \cdots \otimes \bar{a}_{m'+n'}).$$

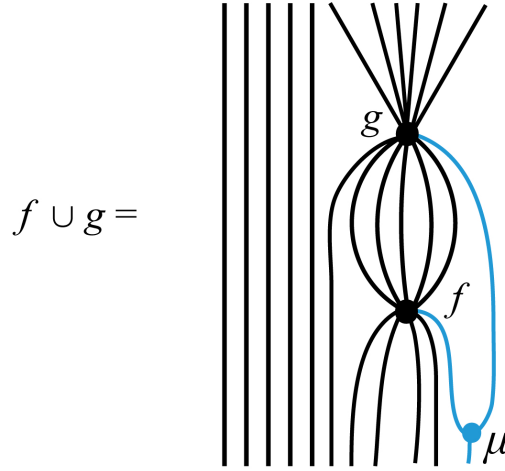


FIGURE 5. The diagram above represents the cup product  $\cup : \mathcal{C}_{\text{sg}}^*(A, A) \otimes \mathcal{C}_{\text{sg}}^*(A, A) \rightarrow \mathcal{C}_{\text{sg}}^*(A, A)$ . The blue edges denote elements in  $A$  while black edges denote elements in  $s\bar{A}$ . We have represented composition by stacking corollas, where  $f$  and  $g$  are represented as corollas as in Figure 1.

From a direct computation, it follows that  $\cup$  equals  $\cup'$  up to homotopy, hence we have

$$\cup' = \cup : \text{HH}^m(A, \Omega^p(A)) \otimes \text{HH}^n(A, \Omega^q(A)) \rightarrow \text{HH}^{m+n}(A, \Omega^{p+q}(A))$$

for any  $m, n \in \mathbb{Z}$  and  $p, q \in \mathbb{Z}_{\geq 0}$ . Therefore, the product  $\cup'$  defines a product on the colimit  $\varinjlim_{p \in \mathbb{Z}_{\geq 0}} \text{HH}^*(A, \Omega^p(A))$ , which corresponds to the cup product  $\cup$  on  $H^*(\mathcal{C}_{\text{sg}}^*(A, A))$  under the canonical isomorphism

$$H^*(\mathcal{C}_{\text{sg}}^*(A, A)) \cong \varinjlim_{p \in \mathbb{Z}_{\geq 0}} \text{HH}^*(A, \Omega^p(A)).$$

On the other hand, it is clear that the Yoneda products are compatible with the morphism  $\chi : H^*(\mathcal{C}_{\text{sg}}^*(A, A)) \rightarrow \text{HH}_{\text{sg}}^*(A, A)$  since  $\chi$  is induced from the triangulated functor  $\mathcal{D}^b(A \otimes A^{\text{op}}) \rightarrow \mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})$ .  $\square$

A tree diagram for the cup product  $\cup$  is given in Figure 5.

**4.2. DGLA structure on  $\mathcal{C}_{\text{sg}}^*(A, A)$ .** Let  $f \in C^{m,*}(A, \Omega^p(A))$  and  $g \in C^{n,*}(A, \Omega^q(A))$ , we define a bracket  $\{f, g\} \in C^{m+n-1,*}(A, \Omega^{p+q}(A))$  as

$$\{f, g\} = f \bullet g - (-1)^{(\deg(f)+1)(\deg(g)+1)} g \bullet f$$

where we denote

$$f \bullet g = \sum_{i=1}^m f \bullet_i g + \sum_{i=1}^p f \bullet_{-i} g$$

and  $f \bullet_i g$  is defined as

$$f \bullet_i g := \begin{cases} (\text{id}^{\otimes q} \otimes f) \circ (\text{id}^{i-1} \otimes (\text{id}^{\otimes n} \otimes \pi) g \otimes \text{id}^{\otimes m-i}) & \text{for } i \geq 1, \\ (\text{id}^{\otimes p+i} \otimes (\text{id}^{\otimes n} \otimes \pi) g \otimes \text{id}^{\otimes -i}) \circ (\text{id}^{\otimes n-1} \otimes f) & \text{for } i \leq -1, \end{cases}$$

where  $\circ$  denotes composition of morphisms and  $\pi : A \rightarrow s\bar{A}$  is the natural projection map. It follows from a direct calculation that the bracket is compatible with the colimit construction, thus it is well-defined on  $\mathcal{C}_{\text{sg}}^*(A, A)$ .

**Proposition 4.2.** *The singular Hochschild complex  $\mathcal{C}_{\text{sg}}^*(A, A)$ , equipped with the Lie bracket  $\{\cdot, \cdot\}$  is a DGLA.*

*Proof.* The proof is analogous to the one of [Wan1, Proposition 4.6].  $\square$

**Remark 4.3.** The cup product  $\cup$  on  $\mathcal{C}_{\text{sg}}^*(A, A)$  is graded commutative up to homotopy, namely, for  $f \in \mathcal{C}_{\text{sg}}^m(A, A)$  and  $g \in \mathcal{C}_{\text{sg}}^n(A, A)$ ,

$$f \cup g - (-1)^{mn} g \cup f = \delta(f) \bullet g \pm \delta(f \bullet g) \pm f \bullet \delta(g).$$

Hence,  $\cup$  defines a graded commutative associative algebra structure on the cohomology  $H^*(\mathcal{C}_{\text{sg}}^*(A, A))$ . Moreover, we have the following result.

**Theorem 4.4.**  *$(H^*(\mathcal{C}_{\text{sg}}^*(A, A)), \cup, \{\cdot, \cdot\})$  is a Gerstenhaber algebra.*

*Proof.* The proof is analogous to the one of [Wan1, Proposition 4.9].  $\square$

## 5. PRODUCTS AND BV OPERATOR ON THE TATE-HOCHSCHILD COMPLEX $\mathcal{D}^*(A, A)$

We now give explicit formulae for three product structures,  $\star$ ,  $\bullet$ , and  $[\cdot, \cdot]$ , and an operator  $\tilde{\Delta}$  on the Tate-Hochschild complex  $\mathcal{D}^*(A, A)$ . These operations extend some of the algebraic string operations described in [Abb] and [TrZe]. In Section 6, we relate these structures to the dga and dglA structures of  $\mathcal{C}_{\text{sg}}^*(A, A)$  defined in Section 4. Diagrams representing the formulae defining  $\star$  may be found in the Appendix.

**5.1.  $\star$ -product on  $\mathcal{D}^*(A, A)$ .** We will define a product of degree zero

$$\star : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^*(A, A)$$

for a dg symmetric Frobenius algebra  $A$ . This  $\star$ -product extends the Hochschild cup product  $\cup$  in  $C^*(A, A)$ , and the cap product  $\cap$  between  $C^*(A, A)$  and  $C_*(A, A)$ , to a product on the chain complex  $(\mathcal{D}^*(A, A), \delta)$  which is compatible with  $\delta$ , namely,  $\delta$  is a derivation of  $\star$ . Recall that we denote  $\text{Hom}_k((s\bar{A})^{\otimes m}, A)^{m+p}$  by  $C^{m,p}(A, A)$  and  $(A \otimes (s\bar{A})^{\otimes m})^{-m+p}$  by  $C_{-m,p}(A, A)$ . Define the  $\star$ -product on  $\mathcal{D}^*(A, A)$  by the following formulae.

(1) For any  $f \in C^{m,*}(A, A)$  and  $g \in C^{n,*}(A, A)$ ,

$$f \star g := f \cup g \in C^{m+n,*}(A, A)$$

where

$$f \cup g(\bar{a}_1 \otimes \cdots \otimes \bar{a}_{m+n}) = (-1)^{\deg(g)\epsilon_m} f(\bar{a}_1 \otimes \cdots \otimes \bar{a}_m) g(\bar{a}_{m+1} \otimes \cdots \otimes \bar{a}_{m+n}),$$

and  $\epsilon_m = \sum_{i=1}^m \deg(a_i) - m$ . Namely, this is the usual cup product in  $C^*(A, A)$ .

(2) For any  $f \in C^{m,*}(A, A)$  and  $\alpha = \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes a_{n+1} \in C_{-n,*}(A, A)$

(a) if  $m - n > 0$ , we define  $\alpha \star f, f \star \alpha \in C^{m-n-1,*}(A, A)$  as follows: For any  $\bar{b}_1 \otimes \cdots \otimes \bar{b}_{m-n-1} \in (s\bar{A})^{\otimes m-n-1}$ ,

$$\alpha \star f(\bar{b}_1 \otimes \cdots \otimes \bar{b}_{m-n-1}) := \sum_i (-1)^{\kappa_i} e_i f(\bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes \overline{a_{n+1} f_i} \otimes \bar{b}_1 \otimes \cdots \otimes \bar{b}_{m-n-1})$$

$$f \star \alpha(\bar{b}_1 \otimes \cdots \otimes \bar{b}_{m-n-1}) := \sum_i (-1)^{\lambda_i} f(\bar{b}_1 \otimes \cdots \otimes \bar{b}_{m-n-1} \otimes \bar{e}_i \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) a_{n+1} f_i,$$

where the signs are given by

$$\kappa_i = \deg(f_i) \deg(\alpha) + \deg(e_i) + \deg(\alpha) - \deg(a_{n+1}) + (\deg(\alpha) + \deg(f_i) - 1) \deg(f),$$

and

$$\lambda_i = \deg(\alpha)\deg(f_i) + (\deg(\alpha) + k - 1) \left( \sum_{j=1}^{m-n-1} \deg(b_j) - m + n + 1 \right)$$

(b) if  $m - n \leq 0$  define  $\alpha \star f, f \star \alpha \in C_{-m-n,*}(A, A)$  by

$$\begin{aligned} \alpha \star f &:= (-1)^{\epsilon_m(\deg(\alpha) - \epsilon_m + \deg(f))} \overline{a_{m+1}} \otimes \cdots \otimes \overline{a_n} \otimes a_{n+1} f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) \\ f \star \alpha &:= (-1)^{\epsilon_{n-m}\deg(f)} \overline{a_1} \otimes \cdots \otimes \overline{a_{n-m}} \otimes f(\overline{a_{n-m+1}} \otimes \cdots \otimes \overline{a_n}) a_{n+1}, \end{aligned}$$

where  $\epsilon_l = \sum_{j=1}^l \deg(a_j) - l$  as before.

(3) For any  $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes a_{n+1} \in C_{-n,*}(A, A)$  and  $\beta = \overline{b_1} \otimes \cdots \otimes \overline{b_m} \otimes b_{m+1} \in C_{-m,*}(A, A)$  define  $\beta \star \alpha \in C_{-m-n-1,*}(A, A)$  by

$$\beta \star \alpha := \sum_i (-1)^{\gamma_i} \overline{a_1} \otimes \cdots \otimes \overline{a_{n+1}e_i} \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_m} \otimes b_{m+1} f_i,$$

where  $\gamma_i = \deg(\beta)\deg(\alpha) + (\deg(\alpha) + \deg(\beta))\deg(f_i) + \deg(\alpha)\deg(e_i) + \deg(\alpha) - \deg(a_{n+1})$ .

**Remark 5.1.** When  $A$  is commutative the formula given in (3) for  $\beta \star \alpha$  may be written as

$$(12) \quad \sum_i \pm \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes \overline{(a_{n+1}b_{m+1})'} \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_m} \otimes (a_{n+1}b_{m+1})'',$$

where we have written  $\Delta : A \rightarrow A \otimes A$  as  $\Delta(x) = \sum x' \otimes x''$ . Formula (12) agrees with a product of degree  $k - 1$  described in [Abb] and [Kla]. This operation does *not* define a chain map on the Hochschild chain complex. In fact, this formula yields a chain map between two products  $\ast_0$  and  $\ast_1$ , as described in [Abb]. When  $A$  is commutative, the product induces a chain map on the subcomplex  $\bigoplus_{i \geq 1}^\infty (s\overline{A})^{\otimes i} \otimes A$  of  $C_{*,*}(A, A)$  as proposed by [Abb]. Another way to obtain a chain map in the commutative case is to modify  $\star$  by defining a new product  $\alpha \tilde{\star} \beta := (\alpha - p(\beta)) \star (\beta - p(\beta))$ , where  $p : C_{*,*}(A, A) \rightarrow \mathbb{K} \otimes A$  is the natural projection map, as proposed by [Kla]. However, we do not assume commutativity in our setting.

**Remark 5.2.** The associativity relation holds strictly for three elements in  $C_{*,*}(A, A)$  or three elements in  $C^{*,*}(A, A)$ . However, in general, for three mixed elements associativity holds up to homotopy, as we will see later. Furthermore, we will show that there is an  $A_\infty$ -algebra structure on  $\mathcal{D}^*(A, A)$  extending the differential  $\delta$  and the product  $\star$  (cf. Theorem 6.3 below).

We have a non-degenerated pairing  $\langle \cdot, \cdot \rangle$  between  $C^{m,*}(A, A)$  and  $C_{-m,*}(A, A)$  for any  $m \in \mathbb{Z}_{\geq 0}$ . Note that when  $m = 0$ , it is exactly defined by the inner product of  $A$ . For  $m > 0$ ,  $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} \in C_{-m,*}(A, A)$  and  $f \in C^{m,*}(A, A)$ , define

$$\langle f, \alpha \rangle := \langle f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}), a_{m+1} \rangle$$

and

$$\langle \alpha, f \rangle := (-1)^{\epsilon_m(\deg(a_{m+1}) + \deg(f))} \langle a_{m+1}, f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) \rangle$$

where  $\epsilon_m = \sum_{i=1}^m \deg(a_i) - m$ . Note that with these definitions we have  $\langle f, \alpha \rangle = (-1)^{\deg(\alpha)\deg(f)} \langle \alpha, f \rangle$ . We may extend this pairing to  $\mathcal{D}^*(A, A)$  by defining  $\langle f, g \rangle = 0 = \langle \alpha, \beta \rangle$  for any  $f, g \in C^{*,*}(A, A)$  and  $\alpha, \beta \in C_{*,*}(A, A)$  and  $\langle \alpha, h \rangle = 0 = \langle h, \alpha \rangle$  for any  $\alpha \in C_{-m,*}(A, A)$  and  $h \in C^{n,*}(A, A)$  with  $m \neq n$ .

**Lemma 5.3.** *The pairing is compatible with the differential in  $\mathcal{D}^*(A, A)$ , namely, we have  $\langle \delta x, y \rangle = \langle x, \delta y \rangle$  for any  $x, y \in \mathcal{D}^*(A, A)$ .*

*Proof.* This is a straightforward computation.  $\square$

**Lemma 5.4.** *The  $\star$ -product is compatible with the pairing  $\langle \cdot, \cdot \rangle$ . Namely, for any  $x, y, z \in \mathcal{D}^*(A, A)$  we have*

$$\langle x \star y, z \rangle = \langle x, y \star z \rangle.$$

*Proof.* Let  $f \in C^{m,*}(A, A)$ ,  $g \in C^{n,*}(A, A)$  and  $\alpha \in C_{-m-n,*}(A, A)$ , then

$$\begin{aligned} \langle \alpha \star f, g \rangle &= \pm \langle \overline{a_{m+1}} \otimes \cdots \otimes \overline{a_{m+1+n}} f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}), g \rangle \\ &= \pm \langle a_{m+1+n} f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}), g(\overline{a_{m+1}} \otimes \cdots \otimes \overline{a_{m+1+n}}) \rangle \\ &= \pm \langle a_{m+1+n}, f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) g(\overline{a_{m+1}} \otimes \cdots \otimes \overline{a_{m+1+n}}) \rangle \\ &= \pm \langle a_{m+1+n}, f \star g(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+n}}) \rangle \\ &= \langle \alpha, f \star g \rangle. \end{aligned}$$

A similar calculation yields  $\langle f \star \alpha, g \rangle = \langle f, \alpha \star g \rangle$ .

Now suppose  $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} \in C_{-m,*}(A, A)$ ,  $\beta = \overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes b_{n+1} \in C_{-n,*}(A, A)$  and  $f \in C^{m+n+1,*}(A, A)$ . Then

$$\begin{aligned} \langle f \star \alpha, \beta \rangle &= \langle f \star \alpha(\overline{b_1} \otimes \cdots \otimes \overline{b_n}), b_{n+1} \rangle \\ &= \sum_i \pm \langle f(\overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes \overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m}) a_{m+1} f_i, b_{n+1} \rangle \\ &= \sum_i \pm \langle f(\overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes \overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m}), a_{m+1} f_i b_{n+1} \rangle \\ &= \sum_i \pm \langle f(\overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes \overline{b_{n+1} e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m}), a_{m+1} f_i \rangle \\ &= \langle f, \alpha \star \beta \rangle. \end{aligned}$$

A similar computation yields  $\langle \alpha \star f, \beta \rangle = \langle \alpha, f \star \beta \rangle$ .  $\square$

**5.2.  $\bullet$ -product and  $[\cdot, \cdot]$ -bracket on  $\mathcal{D}^*(A, A)$ .** We will define a product

$$\bullet : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^*(A, A)$$

of degree  $-1$  for any dg symmetric Frobenius algebra  $A$ . This product generalizes the Gerstenhaber  $\circ$ -product in  $C^{*,*}(A, A)$  and an analogous product constructed in  $C_{*,*}(A, A)$  (cf. [Abb, Wan1]). We then define a bracket

$$[\cdot, \cdot] : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^*(A, A)$$

of degree  $-1$  as the commutator of the  $\bullet$ -product.

Define the  $\bullet$ -product case by case:

(1) For  $f \in C^{m,*}(A, A)$ ,  $g \in C^{n,*}(A, A)$ , we define  $f \bullet g \in C^{m+n-1,*}(A, A)$  by

$$\begin{aligned} &f \bullet g(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+n-1}}) \\ &= \sum_{i=1}^m (-1)^{(\deg(g)+1)\epsilon_{i-1} + \deg(f)} f(\overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{g(\overline{a_i} \otimes \cdots \otimes \overline{a_{i+n-1}})}) \otimes \overline{a_{i+n}} \otimes \cdots \otimes \overline{a_{m+n-1}}, \end{aligned}$$

where  $\epsilon_l = \sum_{i=1}^l \deg(a_i) - l$ . We will use the same definition for  $\epsilon_l$  below.

- (2) For  $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} \in C_{-r,*}(A, A)$  and  $\beta = \overline{b_1} \otimes \cdots \otimes \overline{b_s} \otimes b_{s+1} \in C_{-s,*}(A, A)$  define  $\alpha \bullet \beta \in C_{-r-s-2,*}(A, A)$  by

$$\alpha \bullet \beta = \sum_i \sum_{j=1}^{r+1} (-1)^{\gamma_{ij}} \overline{a_1} \otimes \cdots \otimes \overline{a_{j-1}} \otimes \overline{e_i} \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_{s+1}} \overline{f_i} \otimes \overline{a_j} \otimes \cdots \otimes a_{r+1},$$

where  $\gamma_{ij} = \deg(\beta)(\deg(\alpha) - \epsilon_{j-1}) + (\deg(f_i) - 1)(\epsilon_{j-1} + \deg(\beta)) + \epsilon_{j-1} \deg(e_i) - 1 + \deg(\beta) - \deg(b_{s+1})$ .

- (3) Let  $f \in C^{m,*}(A, A)$  and  $\alpha \in C_{-r,*}(A, A)$ .

(a) If  $m \geq r + 2$ , define  $f \bullet \alpha \in C^{m-r-2,*}(A, A)$  by

$$f \bullet \alpha := \sum_{i,j} \sum_{l=0}^r (-1)^{\gamma_{ijl}} \langle f(\overline{a_1} \otimes \cdots \otimes \overline{a_l} \otimes \overline{e_i} \otimes \text{id}^{\otimes m-r-2} \otimes \overline{e_j} \otimes \overline{a_{l+1}} \otimes \cdots \otimes \overline{a_r}), a_{r+1} \rangle f_i f_j,$$

where  $\gamma_{ijl} = k + (\deg(f) + \deg(\alpha)) \deg(f_j) + (\deg(e_j) - 1 + \deg(f) + \deg(\alpha)) \deg(f_i) + (\deg(f) + \epsilon_l)(\deg(e_i) + \deg(e_j) - 2)$  and we interpret the symbol  $\text{id}^{\otimes m-r-2}$  denotes an empty slot where we plug in an element of  $(s\overline{A})^{\otimes m-r-2}$ . Similarly, define

$$\alpha \bullet f := \sum_i \sum_{j=1}^{m-r-1} (-1)^{\eta_i} f(\text{id}^{\otimes j-1} \otimes \overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{r+1}} \overline{f_i} \otimes \text{id}^{\otimes m-r-j-1})$$

where  $\eta_i = \deg(f)(k - 2 + \deg(\alpha)) + (\deg(f_i) - 1) \deg(\alpha) + \deg(f) - 1 + \deg(\alpha) - \deg(a_{r+1})$ .

(b) If  $m < r + 2$ , define  $f \bullet \alpha \in C_{-(r-m+1),*}(A, A)$  by

$$f \bullet \alpha := \sum_{i=0}^{r-m} (-1)^{\epsilon_i(\deg(f)+1)} \overline{a_1} \otimes \cdots \otimes \overline{f(\overline{a_{i+1}} \otimes \cdots \otimes \overline{a_{i+m}})} \otimes \overline{a_{i+m+1}} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1},$$

and similarly

$$\alpha \bullet f := \sum_j \sum_{i=1}^m (-1)^{\rho_{ij}} \overline{a_i} \otimes \cdots \otimes \overline{a_{i+r-m}} \otimes e_j \langle f(\overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{f_j} \otimes \overline{a_{i+r-m+1}} \otimes \cdots \otimes \overline{a_r}), a_{r+1} \rangle,$$

where  $\rho_{ij} = \deg(e_j) + (\sum_{l=i}^{i+r-m} \deg(a_l) - r + m - 1) \epsilon_{i-1} + \epsilon_{i+r-m}(\deg(f_j) - 1) + (\sum_{l=i}^{i+r-m} \deg(a_l) - r + m - 1) \deg(e_j) + \deg(f)(\epsilon_{i-1} + \deg(f_i) - 1 + \deg(\alpha) - \epsilon_{i+r-m})$ .

**Lemma 5.5.** *For any  $\alpha, \beta, \gamma \in \mathcal{D}^*(A, A)$ , we have*

$$\langle \alpha \bullet \beta, \gamma \rangle = \langle \alpha, \beta \bullet \gamma \rangle.$$

*Proof.* This is clear by direct computation. In fact, the definitions of  $f \bullet \alpha$  in case (3a) above and  $\alpha \bullet f$  in case (3b) above are determined by the definition of  $\bullet$  in the other cases, the non-degeneracy of the pairing, and the equation  $\langle \alpha \bullet \beta, \gamma \rangle = \langle \alpha, \beta \bullet \gamma \rangle$ . This is how the formula for  $\bullet$  in these two cases was obtained. For the other cases the formula is given canonically by generalizing the Gerstenhaber's classical  $\circ$ -product.  $\square$

**Definition 5.6.** The bracket of degree  $-1$

$$[\cdot, \cdot] : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^*(A, A)$$

is defined as

$$[x, y] := x \bullet y - (-1)^{(|x|-1)(|y|-1)} y \bullet x.$$

Thus it follows from Lemma 5.5 that

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle.$$

**Lemma 5.7.** *The bracket  $[\cdot, \cdot]$  is compatible with the differential  $\delta$  in  $\mathcal{D}^*(A, A)$ . Namely, we have*

$$(13) \quad \delta([x, y]) = [\delta(x), y] + (-1)^{|x|-1} [x, \delta(y)]$$

for any  $x, y \in \mathcal{D}^*(A, A)$ . As a consequence, the bracket  $[\cdot, \cdot]$  is well-defined on  $H^*(\mathcal{D}^*(A, A))$ .

*Proof.* This follows from Lemma 5.3 and the fact that Identity (13) holds for either  $x, y \in C^{\geq 0, *}(A, A)$  or  $x, y \in C_{< 0, *}(A, A)$ .  $\square$

**Remark 5.8.** In general, the bracket  $[\cdot, \cdot]$  on  $\mathcal{D}^*(A, A)$  does not satisfy the Jacobi identity. However, in Section 6 we will see that it does on cohomology  $H^*(\mathcal{D}^*(A, A))$ .

**5.3. BV operator  $\tilde{\Delta}$  on  $\mathcal{D}^*(A, A)$ .** In this subsection we extend the Connes' boundary operator  $B$  in the Hochschild chain complex  $C_*(A, A)$  to the Tate-Hochschild complex  $\mathcal{D}^*(A, A)$ . Recall that Connes' boundary operator  $B : C_{m, *}(A, A) \rightarrow C_{m+1, *}(A, A)$  sends a monomial element  $x := \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} \in C_{m, *}(A, A)$  to

$$B(x) := \sum_{i=1}^{m+1} (-1)^{\epsilon_m + \epsilon_{i-1} \deg(\alpha)} \overline{a_i} \otimes \cdots \otimes \overline{a_m} \otimes \overline{a_{m+1}} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes 1,$$

where  $\epsilon_l = \sum_{i=1}^l \deg a_i - l$ . It is well-known that  $B \circ B = 0$  and  $B$  is compatible with the differential of  $C_*(A, A)$  (cf. e.g. [Lod]). Thus  $B$  induces a differential in Hochschild homology  $\mathrm{HH}_*(A, A)$

$$B : \mathrm{HH}_*(A, A) \rightarrow \mathrm{HH}_{*+1}(A, A).$$

The pairing  $\langle \cdot, \cdot \rangle$  of  $A$  induces a linear isomorphism  $C_{m, *}(A, A)^\vee \cong C^{m, *}(A, A)$  for each  $m \in \mathbb{Z}_{\geq 0}$ , thus there is an isomorphism between  $C_*(A, A)^\vee$ . We may use this isomorphism, to define

$$\Delta : C^*(A, A) \rightarrow C^{*-1}(A, A)$$

by

$$\langle \Delta(f), \overline{a_1} \otimes \cdots \otimes \overline{a_{m-1}} \otimes a_m \rangle := (-1)^{\deg(f)} \langle f, B(\overline{a_1} \otimes \cdots \otimes \overline{a_{m-1}} \otimes a_m) \rangle.$$

From  $B \circ B = 0$  and  $B \circ d + d \circ B = 0$ , it follows that  $\Delta \circ \Delta = 0$  and  $\Delta \circ \delta + \delta \circ \Delta = 0$ , thus we have an induced map

$$\Delta : \mathrm{HH}^*(A, A) \rightarrow \mathrm{HH}^{*+1}(A, A)$$

with  $\Delta \circ \Delta = 0$ .

We may combine the Connes' boundary operator  $B$  and its dual  $\Delta$  to obtain an operator

$$\tilde{\Delta} : \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^{*-1}(A, A)$$

defined by

$$\tilde{\Delta}(x) := \begin{cases} \Delta(x) & \text{if } x \in C^{> 0, *}(A, A), \\ 0 & \text{if } x \in C^{0, *}(A, A), \\ B(x) & \text{if } x \in C_{\leq 0, *}(A, A). \end{cases}$$



**Remark 5.9.**  $\tilde{\Delta}$  is compatible with the pairing  $\langle \cdot, \cdot \rangle$  on  $\mathcal{D}^*(A, A)$ , that is to say,

$$\langle \tilde{\Delta}(x), y \rangle = (-1)^{\deg(x)} \langle x, \tilde{\Delta}(y) \rangle$$

for any  $x, y \in \mathcal{D}^*(A, A)$ . It is clear that  $\tilde{\Delta} \circ \tilde{\Delta} = 0$  and  $\tilde{\Delta} \circ \delta + \delta \circ \tilde{\Delta} = 0$ , where  $\delta$  represents the differential of  $\mathcal{D}^*(A, A)$ . So  $\tilde{\Delta}$  induces a differential in the cohomology of  $\mathcal{D}^*(A, A)$

$$\tilde{\Delta} : H^*(\mathcal{D}^*(A, A)) \rightarrow H^{*-1}(\mathcal{D}^*(A, A)).$$

**Proposition 5.10.** *We have the following identity on  $H^*(\mathcal{D}^*(A, A))$*

$$[x, y] = \tilde{\Delta}(x) \star y \pm x \star \tilde{\Delta}(y) \pm \tilde{\Delta}(x \star y),$$

for any  $x, y \in H^*(\mathcal{D}^*(A, A))$ .

*Proof.* It is well-known that such an identity holds for  $[x], [y] \in H^*(\mathcal{D}^*(A, A))$ , where  $[x]$  and  $[y]$  are represented by elements  $x, y \in \mathcal{D}^{\geq 0, *}(A, A)$ ; this follows from the fact that  $\mathrm{HH}^*(A, A)$  is a BV algebra proved in [Kau, Men, Tra]. The identity also holds for  $[x], [y] \in H^*(\mathcal{D}^{*, *}(A, A))$  where  $[x]$  and  $[y]$  are represented by elements  $x, y \in \mathcal{D}^{< 0, *}(A, A)$ . This follows from a computation in [Abb]. Note that Abbaspour's proof of the fact that the *reduced* (shifted by  $1 - k$ ) Hochschild homology  $\widetilde{\mathrm{HH}}_*(A, A)$  is a BV-algebra uses the assumption of graded commutativity of  $A$ . However, the homotopy term  $H(x, y)$  ([Abb, Page 738]) also works here without the commutativity hypothesis if we switch  $a'_0$  and  $a''_0$  in the formula of  $H(x, y)$ . It remains to check the following cases.

- (1) If  $x \in C^{m, *}(A, A)$  and  $y \in C_{n, *}(A, A)$  such that  $m + n \geq 2$ , then, by definition,  $[\cdot, \cdot]$  is determined by the following identity

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle$$

for any  $z \in C_{*, *}(A, A)$ . From the previous argument, it follows that on  $H^*(\mathcal{D}^*(A, A))$ ,

$$[y, z] = \tilde{\Delta}(x) \star y \pm x \star \tilde{\Delta}(y) \pm \tilde{\Delta}(x \star y).$$

So we have that for any  $z \in C_{*, *}(A, A)$ ,

$$\begin{aligned} \langle [x, y], z \rangle &= \langle x, [y, z] \rangle \\ &= \langle x, \tilde{\Delta}(y) \star z \pm y \star \tilde{\Delta}(z) \pm \tilde{\Delta}(y \star z) \rangle \\ &= \langle x \star \tilde{\Delta}(y) \pm \tilde{\Delta}(x \star y) \pm \tilde{\Delta}(x) \star y, z \rangle \end{aligned}$$

where the third identity follows from the compatibility of  $\tilde{\Delta}$  with the pairing  $\langle \cdot, \cdot \rangle$  (cf. Remark 5.9). It follows that

$$[x, y] = \tilde{\Delta}(x) \star y \pm x \star \tilde{\Delta}(y) \pm \tilde{\Delta}(x \star y).$$

- (2) For the remaining cases we may use the same argument as above to check the desired identity.

□

## 6. $A_\infty$ -ALGEBRA AND $L_\infty$ -ALGEBRA STRUCTURES ON $\mathcal{D}^*(A, A)$

**6.1. Homotopy transfer theorem.** We recall the Homotopy Transfer Theorem (cf. Theorem 6.1) and use it to obtain  $A_\infty$ -algebra and  $L_\infty$ -algebra structures on  $\mathcal{D}^*(A, A)$ . Then we compare these transferred structures on  $\mathcal{D}^*(A, A)$  to the  $\star$  and  $[\cdot, \cdot]$  operations defined in Section 5 above.

**Theorem 6.1** (Homotopy Transfer Theorem). *Let  $(V, d_V)$  be a (strong) homotopy retract of  $(W, d_W)$ , namely we have*

$$(14) \quad (V, d_V) \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\Pi} \end{array} (W, d_W) \quad \begin{array}{c} \circlearrowright \\ h \end{array}$$

such that

$$\text{id}_W - \iota \circ \Pi = d_W h + h d_W$$

and

$$\Pi \circ \iota = \text{id}_V.$$

Let  $\mathcal{P}$  be a Koszul operad. Then any  $\mathcal{P}_\infty$ -algebra structure on  $W$  can be transferred into a  $\mathcal{P}_\infty$ -algebra structure on  $V$  such that  $\iota$  extends to an  $\infty$ -quasi-isomorphism.

**Remark 6.2.** We denote the operad encoding associative algebras by  $\text{Ass}$  and the operad encoding Lie algebras by  $\text{Lie}$ . It is well-known that these two operads are both Koszul (cf. e.g. [GiKa, LoVa]). Recall that, in the case when  $W$  has a dga structure then the transferred  $A_\infty$ -algebra structure on  $V$  consists of maps  $m_k : V^{\otimes k} \rightarrow V$  for  $k = 1, 2, 3, \dots$  where each  $m_k$  is given by the sum over all possible trivalent planar rooted trees with  $k$  leaves and each tree is labeled by placing  $\iota$  on the leaves,  $\pi$  on the root,  $h$  on the internal edges, and the product of  $W$  in the (internal) vertices. For more details on the Homotopy Transfer Theorem, we refer to [Kad, LoVa].

**Theorem 6.3.** *There is an  $A_\infty$ -algebra structure  $(m_1, m_2, \dots)$  and an  $L_\infty$ -algebra structure  $(l_1, l_2, \dots)$  on  $\mathcal{D}^*(A, A)$  such that*

- (1)  $m_1 = \delta, m_2 = \star$ , on  $\mathcal{D}^*(A, A)$
- (2)  $l_1 = \delta$ , and  $l_2 = [\cdot, \cdot]$  on  $H^*(\mathcal{D}^*(A, A))$

Furthermore,  $(\mathcal{D}^*(A, A), m_1, m_2, \dots)$  and  $(\mathcal{D}^*(A, A), l_1, l_2, \dots)$  are  $A_\infty$ -quasi-isomorphic and  $L_\infty$ -quasi-isomorphic to the dga and dgla structures on  $\mathcal{C}_{\text{sg}}^*(A, A)$ , respectively.

*Proof.* It follows from the Homotopy Transfer Theorem that we may transfer the dga structure on  $\mathcal{C}_{\text{sg}}^*(A, A)$ , along the homotopy retract discussed in Subsection 3.4, to an  $A_\infty$ -quasi-isomorphic  $A_\infty$ -algebra structure  $(\delta, m_2, \dots)$  on  $\mathcal{D}^*(A, A)$ . Similarly, we may transfer the dgla structure on  $\mathcal{C}_{\text{sg}}^*(A, A)$  to an  $L_\infty$ -quasi-isomorphic  $L_\infty$ -algebra structure  $(\delta, l_2, \dots)$  on  $\mathcal{D}^*(A, A)$ . We will verify that  $m_2 = \star$  at the chain level and  $l_2 = [\cdot, \cdot]$  on cohomology. We check this case by case.

- (1) if  $f, g \in C^{*,*}(A, A)$  then it is straightforward from the definitions that  $m_2(f, g) = \Pi(\iota(f) \cup \iota(g)) = f \cup g$ .
- (2) Let  $\alpha = \overline{a_1} \otimes \dots \otimes \overline{a_m} \otimes a_{m+1} \in C_{-m,*}(A, A)$  and  $\beta = \overline{b_1} \otimes \dots \otimes \overline{b_n} \otimes b_{n+1} \in C_{-n,*}(A, A)$ , where  $m, n \in \mathbb{Z}_{\geq 0}$ , then by the formulae for the transferred structure given in the proof of the Homotopy Transfer Theorem as recalled in Remark 6.2, we have that

$$\begin{aligned} m_2(\beta, \alpha) &= \Pi(\iota(\beta) \cup \iota(\alpha)) \\ &= \pm \sum_{i,j} (\epsilon \otimes \text{id})(\overline{e_i} \otimes \overline{a_1} \otimes \dots \otimes \overline{a_m} \otimes \overline{e_j} \otimes \overline{b_1} \otimes \dots \otimes b_{n+1} f_j a_{m+1} f_i) \\ &= \pm \sum_j \overline{a_1} \otimes \dots \otimes \overline{a_m} \otimes \overline{a_{m+1} e_j} \otimes \overline{b_1} \otimes \overline{b_n} \otimes b_{n+1} f_j \\ &= \beta \star \alpha. \end{aligned}$$

- (3) If  $\alpha \in C_{-m,*}(A, A)$  and  $f \in C^{n,*}(A, A)$  where  $m, n \in \mathbb{Z}_{\geq 0}$ , then

(a) if  $m < n$ , then we have that  $m_2(\alpha, f) \in C^{m-m-1,*}(A, A)$ , moreover

$$\begin{aligned}
m_2(\alpha, f) &= \Pi(\iota(\alpha) \cup \iota(f)) \\
&= \sum_i \pm (\pi_{n-m-1,0} \circ \cdots \circ \pi_{n,m+1})(\overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes a_{m+1} f_i f(?)) \\
&= \sum_i \pm (\pi_{n-m-1,0} \circ \cdots \circ \pi_{n-1,m})(\overline{e_i a_1} \otimes \overline{a_2} \otimes \cdots \otimes a_{m+1} f(\overline{f_i} \otimes ?)) \\
&= \sum_i \pm (\pi_{n-m-1,0} \circ \cdots \circ \pi_{n-2,m-1})(\overline{e_i a_2} \otimes \overline{a_3} \otimes \cdots \otimes a_{m+1} f(\overline{a_1} \otimes \overline{f_i} \otimes ?)) \\
&= \cdots \\
&= \sum_i \pm e_i f(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+1} f_i} \otimes ?) \\
&= \alpha \star f
\end{aligned}$$

where we wrote  $\alpha := \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1}$  and the question mark ? just means a slot to plug in any monomial of length determined by the fact that  $f \in C^{n,*}(A, A)$ . Similarly, we have

$$\begin{aligned}
m_2(f, \alpha) &= \Pi(\iota(f) \cup \iota(\alpha)) \\
&= \sum_i \pm \Pi(f(? \otimes \overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m}) a_{m+1} f_i) \\
&= f \star \alpha
\end{aligned}$$

(b) if  $m \geq n$ , then we have that  $m_2(\alpha, f) \in C_{-(m-n),*}(A, A)$ , thus

$$\begin{aligned}
m_2(\alpha, f) &= \Pi(\iota(\alpha) \cup \iota(f)) \\
&= \sum_i \pm (\pi_{0,m-n+1} \circ \cdots \circ \pi_{n,m+1})(\overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes a_{m+1} f_i f(?)) \\
&= \sum_i \pm (\pi_{0,m-n+1} \circ \cdots \circ \pi_{n-1,m})(\overline{e_i a_1} \otimes \cdots \otimes a_{m+1} f(f_i \otimes ?)) \\
&= \sum_i \pm (\pi_{0,m-n+1} \circ \cdots \circ \pi_{n-2,m-1})(\overline{e_i a_2} \otimes \cdots \otimes a_{m+1} f(\overline{a_1} \otimes \overline{f_i} \otimes ?)) \\
&= \cdots \\
&= \sum_i \pm \pi_{0,m-n-1}(\overline{e_i a_n} \otimes \overline{a_{n+1}} \otimes \cdots \otimes a_{m+1} f(\overline{a_1} \otimes \cdots \otimes \overline{a_{n-1}} \otimes \overline{f_i})) \\
&= \pm \overline{a_{n+1}} \otimes \cdots \otimes a_{m+1} f(\overline{a_1} \otimes \cdots \otimes \overline{a_n}) \\
&= \alpha \star f.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
m_2(f, \alpha) &= \Pi(\iota(f) \cup \iota(\alpha)) \\
&= \sum_i \pm (\pi_{0,m-n+1} \circ \cdots \circ \pi_{n,m+1})(\overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes a_{m+1} f_i f(?)) \\
&= \pm \overline{a_1} \otimes \cdots \otimes \overline{a_{m-n}} \otimes f(\overline{a_{m-n+1}} \otimes \cdots \otimes \overline{a_m}) a_{m+1} \\
&= f \star \alpha.
\end{aligned}$$

Therefore, we have verified that  $m_2 = \star$ .

It remains to verify  $l_2 = [\cdot, \cdot]$  on  $H^*(\mathcal{D}^*(A, A))$ . Again, it follows from the proof of the Homotopy Transfer Theorem that we have the following formulae for  $l_2$ .

(1) If either  $\alpha, \beta \in C_*(A, A)$  or  $\alpha, \beta \in C^*(A, A)$ , then

$$l_2(\alpha, \beta) = [\alpha, \beta].$$

(2) If  $\alpha \in C_{-r,*}(A, A)$  and  $f \in C^{m,*}(A, A)$  such that  $m \geq r + 2$ , then

$$(15) \quad \begin{aligned} & l_2(\alpha, f)(\overline{b_1} \otimes \cdots \otimes \overline{b_n}) = \\ & \sum_{i=1}^{r+1} \pm e_j f(\overline{a_i} \otimes \cdots \otimes \overline{a_{r+1}} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{f_j} \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_n}) + \\ & \sum_{i=1}^n \pm f(\overline{b_1} \otimes \cdots \otimes \overline{b_{i-1}} \otimes \overline{e_j} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{r+1}} \otimes \overline{f_j} \otimes \overline{b_i} \otimes \cdots \otimes \overline{b_n}) + \\ & \sum_{i=1}^{r+1} \pm \langle f(\overline{a_i} \otimes \cdots \otimes \overline{a_r} \otimes \overline{e_j} \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes \overline{e_k} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}}), 1 \rangle f_j a_{r+1} f_k, \end{aligned}$$

where  $n = m - r - 2$  and we wrote  $\alpha := \overline{a_1} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1}$ . Define  $H(\alpha, f) \in C^{m-r-2,*}(A, A)$  as follows,

$$\begin{aligned} H(\alpha, f)(\overline{b_1} \otimes \cdots \otimes \overline{b_{m-r-3}}) &= \sum_{i=1}^{r+1} \sum_{j=0}^{i-1} \sum_{k,l} \\ &\pm \langle f(\overline{a_i} \otimes \cdots \otimes \overline{a_{r+1}} \otimes \cdots \otimes \overline{a_j} \otimes \overline{e_k} \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_{m-r-3}} \otimes \overline{e_l} \otimes \overline{a_{j+1}} \otimes \cdots \otimes \overline{a_{i-1}}), 1 \rangle f_k f_l. \end{aligned}$$

Then by direct computation, we have the following identity,

$$l_2(\alpha, f) - [\alpha, f] = \delta \circ H + H \circ \delta,$$

thus, we have that on cohomology,

$$l_2(\alpha, f) = [\alpha, f].$$

(3) If  $\alpha \in C_{-r,*}(A, A)$  and  $f \in C^{m,*}(A, A)$  such that  $m < r + 2$ , then

$$(16) \quad \begin{aligned} l_2(\alpha, f) &= \sum_{i=1}^m \pm \langle f(\overline{a_i} \otimes \cdots \otimes \overline{a_{m-1}} \otimes \overline{e_j} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}}), 1 \rangle \overline{a_m} \otimes \cdots \otimes a_{r+1} f_j + \\ &\sum_{i=0}^{r-m} \pm \overline{a_1} \otimes \cdots \otimes \overline{a_i} \otimes \overline{f(\overline{a_{i+1}} \otimes \cdots \otimes \overline{a_{i+m}})} \otimes \overline{a_{i+m+1}} \otimes \cdots \otimes a_{r+1} + \\ &\sum_{i=1}^m \pm \overline{a_i} \otimes \cdots \otimes \overline{a_{r-m+i}} \otimes f(\overline{a_{r-m+i+1}} \otimes \cdots \otimes \overline{a_{r+1}} \otimes \cdots \otimes \overline{a_{i-1}}). \end{aligned}$$

Define  $H'(\alpha, f) \in C_{-(r-m),*}(A, A)$  as follows,

$$\begin{aligned} H'(\alpha, f) &= \sum_{i=1}^{m-1} \sum_{j=1}^i \sum_k \\ &\pm \overline{a_i} \otimes \cdots \otimes \overline{a_{i+r-m+1}} \otimes e_k \langle f(\overline{a_j} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{f_k} \otimes \overline{a_{i+r-m+2}} \otimes \cdots \otimes \overline{a_{r+1}} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{j-1}}), 1 \rangle. \end{aligned}$$

Then by direct computation, we have

$$l_2(\alpha, f) - [\alpha, f] = \delta \circ H' + H' \circ \delta,$$

thus we have that on cohomology

$$l_2(\alpha, f) = [\alpha, f].$$

Therefore, we have verified  $l_2 = [\cdot, \cdot]$  on  $H^*(\mathcal{D}^*(A, A))$ .

□

**Remark 6.4.** The homotopy  $m_3$  for the associativity of  $\star = m_2$  is determined by the following explicit formulae which may be obtained from the recipe for transferring a dga structure along a homotopy retraction.

- (1) If  $f, g, h \in C^{*,*}(A, A)$ , then  $m_3(f, g, h) = 0$ .
- (2) If  $\alpha, \beta, \gamma \in C_{*,*}(A, A)$ , then  $m_3(\alpha, \beta, \gamma) = 0$ .
- (3) If  $\alpha, \beta \in C_{*,*}(A, A)$  and  $f \in C^{*,*}(A, A)$ , then  $m_3(\alpha, \beta, f) = 0 = m_3(f, \alpha, \beta)$ .
- (4) If  $\alpha \in C_{*,*}(A, A)$  and  $f, g \in C^{*,*}(A, A)$ , then  $m_3(f, g, \alpha) = 0 = m_3(\alpha, f, g)$ .
- (5) For  $f \in C^{m,*}(A, A)$ ,  $g \in C^{n,*}(A, A)$  and  $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} \in C_{-r,*}(A, A)$  such that  $r < m + n$ , then  $m_3(f, \alpha, g) \in C^{m-r+n,*}(A, A)$  is defined by

$$m_3(f, \alpha, g) = \sum_i \sum_{j=1}^{\min\{m,n,r\}} (-1)^{\kappa_{ij}} f(\text{id}^{\otimes m-r+j} \otimes \overline{e_i} \otimes \overline{a_j} \otimes \cdots \otimes \overline{a_r}) a_{r+1} g(\overline{a_1} \otimes \cdots \otimes \overline{a_{j-1}} \otimes \overline{f_i} \otimes \text{id}^{\otimes n-j}),$$

where  $\kappa_{ij} = \deg(e_i) + \epsilon_{j-1}(\deg(\alpha) - \epsilon_{j-1}) + \deg(f_i)(\deg(\alpha) + \deg(g) + \deg(f)) + \deg(f) \deg(e_i) + \deg(g) \epsilon_{j-1}$  for  $\epsilon_l = \sum_{k=1}^l \deg(a_k) - l$ .

- (6) For  $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} \in C_{r,*}(A, A)$ ,  $\beta = \overline{b_1} \otimes \cdots \otimes \overline{b_s} \otimes b_{s+1} \in C_{s,*}(A, A)$  and  $f \in C^{m,*}(A, A)$  such that  $m \leq r + s$ , then

$$m_3(\alpha, f, \beta) = \sum_i \sum_{j=0}^s (-1)^{\lambda_{ij}} \overline{b_1} \otimes \cdots \otimes \overline{b_j} \otimes \overline{e_i} \otimes \overline{a_{w+1}} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} f(\overline{a_1} \otimes \cdots \otimes \overline{a_w} \otimes \overline{f_i} \otimes \overline{b_{j+1}} \otimes \cdots \otimes \overline{b_s}) b_{s+1}$$

where the symbol  $w$  means  $m - s + j - 1$  and  $\lambda_{ij} = \deg(e_i) + (\sum_{l=1}^j \deg(b_l) - j)(\deg(\alpha) + \deg(f)) + \deg(f_i)(\sum_{l=1}^j \deg(b_l) - j + \deg(\alpha) + \deg(f)) + (\sum_{l=1}^w \deg(a_l) - w) \deg(\alpha) + \deg(e_i)(\sum_{l=1}^j b_l - j) + \deg(f)(\sum_{l=1}^w \deg(a_l) - w)$ .

In case (5) if either  $f$  or  $g$  is an element in  $C^{0,*}(A, A)$  then  $m_3(f, \alpha, g) = 0$ . Similarly, in case (6) if  $f \in C^{0,*}(A, A)$  we have  $m_3(\alpha, f, \beta) = 0$ .

**Proposition 6.5.**  $(\mathcal{D}^*(A, A), (m_1, m_2, \dots), \langle \cdot, \cdot \rangle)$  is a strictly unital (cf. [KoSo, Definition 4.1]) cyclic  $A_\infty$ -algebra with  $m_k = 0$  for  $k \geq 4$ , namely,

$$\langle m_p(\alpha_0 \otimes \cdots \otimes \alpha_{p-1}), \alpha_p \rangle = (-1)^{\deg(\alpha_0)(\deg(\alpha_1) + \cdots + \deg(\alpha_p))} \langle m_p(\alpha_1 \otimes \cdots \otimes \alpha_p), \alpha_0 \rangle$$

for any  $\alpha_0, \dots, \alpha_p \in \mathcal{D}^*(A, A)$ , where  $\langle \cdot, \cdot \rangle$  is the pairing on  $\mathcal{D}^*(A, A)$  (defined in Section 5) induced by the pairing of the dg symmetric Frobenius algebra  $A$ .

*Proof.* To prove that  $m_k = 0$  for  $k \geq 4$  we first prove the following claim.

**Claim 6.6.** We have that the following expressions vanish:

- (1) If  $\alpha_1 \in C^*(A, A)$  then  $h(\iota(\alpha_1) \cup \iota(\alpha_2)) = 0$  for any  $\alpha_2 \in \mathcal{D}^*(A, A)$ .
- (2) If  $\alpha_1, \alpha_2 \in C_*(A, A)$  then  $h(\iota(\alpha_1) \cup \iota(\alpha_2)) = 0$ .
- (3) For any  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{D}^*(A, A)$  we have

$$h(h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup \iota(\alpha_3)) = 0$$

and

$$h(\iota(\alpha_1) \cup h(\iota(\alpha_2) \cup \iota(\alpha_3))) = 0.$$

*Proof of Claim.* Identities (1) and (2) are easy to check. Let us check the first identity in (3), namely  $h(h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup \iota(\alpha_3)) = 0$ . By (1) this identity holds if  $\alpha_1 \in C^*(A, A)$

and by (2) it holds if  $\alpha_1, \alpha_2 \in C_*(A, A)$ . It remains to check it holds when  $\alpha_1 \in C_*(A, A)$  and  $\alpha_2 \in C^*(A, A)$ . In this case,

$$(17) \quad h(\iota(\alpha_1) \cup \iota(\alpha_2)) = \sum_{i=1}^{\min\{m,r\}} \sum_{k_1} \cdots \sum_{k_i} \pm \epsilon(\overline{e_{k_1} x_1}) \cdots \epsilon(\overline{e_{k_{i-1}} x_{i-1}}) \overline{e_{k_i}} \otimes \overline{x_i} \otimes \cdots \otimes x_{m+1} \alpha_2(\overline{f_{k_1}} \otimes \cdots \otimes \overline{f_{k_i}} \otimes \text{id}^{\otimes m-i})$$

where we wrote

$$\alpha_1 := \overline{x_1} \otimes \cdots \otimes \overline{x_m} \otimes x_{m+1}.$$

Then for any  $\alpha_3 \in \mathcal{D}^*(A, A)$ , we have that

$$h(h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup \iota(\alpha_3)) = 0.$$

Indeed, we note that the left most tensor element in each monomial in the sum given by  $h(\iota(\alpha_1) \cup \iota(\alpha_2))$  is  $e_{k_i}$ , so  $h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup \iota(\alpha_3) \in \mathcal{C}_{\text{sg}}^*(A, A)$  is a sum of terms each of which either has  $e_{k_i}$  as the first output (leg) or lies in  $\iota(C_*(A, A))$  otherwise. Note that the action of  $h$  on those terms with  $e_{k_i}$  as the first output is zero because of the fact that  $\sum \epsilon(e_{k_i}) \overline{f_{k_i}} = 0$  in  $\overline{A}$ , and the action of  $h$  on  $\iota(C_*(A, A))$  is zero as well because of dimension reasons. Hence,  $h(h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup \iota(\alpha_3)) = 0$ . The identity

$$h(h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup \iota(\alpha_3)) = 0$$

for any  $\alpha_i \in \mathcal{D}^*(A, A)$  ( $i = 1, 2, 3$ ) follows from a similar argument.  $\square$

From Claim 6.6 above and the Homotopy Transfer Theorem, we have

$$m_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \pm \Pi(h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup h(\iota(\alpha_3) \cup \iota(\alpha_4))).$$

The right hand side of the above equation also vanishes. Indeed, by Claim 6.6, we may assume that  $\alpha_1, \alpha_3 \in C_*(A, A)$  and  $\alpha_2, \alpha_4 \in C^*(A, A)$ . Then from the identity (17) above it follows that  $h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup h(\iota(\alpha_3) \cup \iota(\alpha_4))$  is the sum of terms with the first output  $e_i$ , where  $\sum_i e_i \otimes f_i = \Delta(1)$ . Hence,  $\Pi(h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup h(\iota(\alpha_3) \cup \iota(\alpha_4)))$  is a sum of terms containing  $\sum_i \epsilon(e_i) \overline{f_i} = 0$ , and so  $m_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$ .

In order to check  $m_k = 0$  for  $k > 4$ , because of Claim 6.6 it is sufficient to check the following identity

$$h(h(\iota(\alpha_1) \cup \iota(\alpha_2)) \cup h(\iota(\alpha_3) \cup \iota(\alpha_4))) = 0,$$

for any  $\alpha_i \in \mathcal{D}^*(A, A)$  ( $i = 1, \dots, 5$ ) and the proof of this identity is completely analogous to the one of  $m_4 = 0$  above.

We now check the cyclic compatibilities (or Calabi-Yau conditions). The cyclic compatibility for  $m_2$  was verified in Lemma 5.4. We proceed to verify the identity

$$\langle m_3(\alpha_0, \alpha_1, \alpha_2), \alpha_3 \rangle = \pm \langle m_3(\alpha_1, \alpha_2, \alpha_3), \alpha_0 \rangle$$

for any  $\alpha_i \in \mathcal{D}^*(A, A)$  ( $i = 0, \dots, 3$ ). Based on the formulae for  $m_3$  given in Remark 6.4, it is sufficient to check the case where  $\alpha_0 = \overline{a_1} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} \in C_{-r,*}(A, A)$ ,  $\alpha_2 = \overline{b_1} \otimes \cdots \otimes \overline{b_s} \otimes b_{s+1} \in C_{-s,*}(A, A)$  and  $\alpha_1 \in C^{m,*}(A, A)$ ,  $\alpha_3 \in C^{r+s-m+1,*}(A, A)$ , and  $r + s \geq m$ . We have

$$\begin{aligned} \langle m_3(\alpha_0, \alpha_1, \alpha_2), \alpha_3 \rangle &= \sum_i \sum_{j=1}^{\min\{m,n,s\}} \pm \langle \alpha_3(\overline{b_1} \otimes \cdots \otimes \overline{b_{j-1}} \otimes \overline{e_i} \otimes \overline{a_{m-s+j+1}} \otimes \cdots \otimes \overline{a_r}), a_{r+1} \alpha_1(\overline{a_1} \otimes \cdots \otimes \overline{f_i} \otimes \cdots \otimes \overline{b_s}) b_{s+1} \rangle. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} \langle \alpha_0, m_3(\alpha_1, \alpha_2, \alpha_3) \rangle &= \sum_i \sum_{j=1}^{\min\{m,n,s\}} \\ &\pm \langle a_{r+1}, \alpha_1(\overline{a_1} \otimes \cdots \otimes \overline{a_{m-s+j}} \otimes \overline{e_i} \otimes \overline{b_j} \otimes \cdots \otimes \overline{b_r}) b_{s+1} \alpha_3(\overline{b_1} \otimes \cdots \otimes \overline{b_{j-1}} \otimes \overline{f_i} \otimes \cdots \otimes \overline{a_r}) \rangle. \end{aligned}$$

Therefore, by the compatibility between the product and pairing of  $A$  and by the symmetry of the pairing, we have

$$\langle m_3(\alpha_0, \alpha_1, \alpha_2), \alpha_3 \rangle = \pm \langle \alpha_0, m_3(\alpha_1, \alpha_2, \alpha_3) \rangle,$$

so the cyclic compatibilities hold. On the other hand, the strictly unital condition holds since we have  $m_3(\alpha_1, \alpha_2, \alpha_3) = 0$  if one of the three elements  $\alpha_1, \alpha_2$  and  $\alpha_3$  is  $1 \in C^0(A, A)$ . Therefore we obtain a strictly unital cyclic  $A_\infty$ -algebra structure on  $\mathcal{D}^*(A, A)$ .  $\square$

**Corollary 6.7.** *Let  $A$  be a dg symmetric Frobenius algebra then  $(\mathrm{HH}_{\mathrm{sg}}^*(A, A), \cup, \{\cdot, \cdot\}, \tilde{\Delta})$  is a BV-algebra, where  $\tilde{\Delta}$  is defined in Section 5.3 above.*

*Proof.* From Theorem 4.4 it follows that  $(\mathrm{HH}_{\mathrm{sg}}^*(A, A), \cup, \{\cdot, \cdot\})$  is a Gerstenhaber algebra. It remains to verify (18)

$$(18) \quad \{x, y\} = \tilde{\Delta}(x) \cup y \pm \tilde{\Delta}(x \cup y) \pm x \cup \tilde{\Delta}(y).$$

On the other hand, from Proposition 5.10 it follows that such an identity holds for the Lie bracket  $[\cdot, \cdot]$  on  $H^*(\mathcal{D}^*(A, A))$ , namely,

$$[x, y] = \tilde{\Delta}(x) \star y \pm \tilde{\Delta}(x \star y) \pm x \star \tilde{\Delta}(y).$$

Under the canonical isomorphism  $H^*(\mathcal{D}^*(A, A)) \cong \mathrm{HH}_{\mathrm{sg}}^*(A, A)$ , we have that  $[\cdot, \cdot] = \{\cdot, \cdot\}$  and  $\cup = \star$  from Proposition 6.5, thus Identity (18) holds.  $\square$

**Remark 6.8.** This result was obtained in [EuSc] in the case where  $A$  is an ordinary periodic (i.e.  $A \cong \Omega^n(A)$  in  $\mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$  for some  $n \in \mathbb{Z}_{>0}$ ) symmetric Frobenius algebra and then was generalized to any (ordinary) symmetric Frobenius algebra in [Wan1].

## 7. AN APPLICATION TO STRING TOPOLOGY

Let  $M$  be a simply-connected closed manifold of dimension  $k$ . Let  $C^*(M, \mathbb{K})$  be the singular cochain complex over a field  $\mathbb{K}$ , which is a differential graded associative  $\mathbb{K}$ -algebra. We also denote  $C^*(M, \mathbb{K})$  by  $C^*(M)$  for short, if the base field  $\mathbb{K}$  is fixed. By the main result in [LaSt], there is a cdga  $(A, d)$  and a zig-zag of quasi-isomorphisms between  $(A, d)$  and  $C^*(M)$ .

$$(A, d) \xleftarrow{\cong} \cdots \xrightarrow{\cong} C^*(M)$$

such that  $(A, d)$  is a simply connected commutative differential graded Frobenius algebra of degree  $k$  and, moreover, the induced isomorphism  $H^*(A) \cong H^*(M; \mathbb{K})$  is an isomorphism of Frobenius algebras. We call such  $(A, d)$  a Frobenius cdga-model of  $M$ . Denote by  $LM$  the free loop space of  $M$ , namely,  $LM := \mathrm{Map}(S^1, M)$ . In this section we prove an invariance result for singular Hochschild cohomology and apply it to compute the singular Hochschild cohomology  $\mathrm{HH}_{\mathrm{sg}}^*(C^*(M), C^*(M))$ . More precisely, we will prove the following.

**Theorem 7.1.** *Let  $M$  be a simply-connected closed manifold of dimension  $k$ . Let  $\mathbb{K}$  be a field of characteristic zero. Then we have*

(1) if the Euler characteristic  $\chi(M) = 0$ , then

$$\mathrm{HH}_{\mathrm{sg}}^i(C^*(M), C^*(M)) = \begin{cases} H_{k-i}(LM) & \text{if } i < k-1, \\ H_1(LM) \oplus H^0(LM) & \text{if } i = k-1, \\ H_0(LM) \oplus H^1(LM) & \text{if } i = k, \\ H^{i-k+1}(LM) & \text{if } i > k; \end{cases}$$

(2) if the Euler characteristic  $\chi(M) \neq 0$ , then

$$\mathrm{HH}_{\mathrm{sg}}^i(C^*(M), C^*(M)) = \begin{cases} H_{k-i}(LM) & \text{if } i \leq k-1, \\ H^{i-k+1}(LM) & \text{if } i \geq k. \end{cases}$$

**7.1. Pull-back and push-forward.** Let  $(A, d_1)$  and  $(B, d_2)$  be two differential (non-negatively) graded associative algebras over a field  $\mathbb{K}$ . Let  $f : (A, d_1) \rightarrow (B, d_2)$  be a morphism of dg  $\mathbb{K}$ -algebras. There is a pull-back functor

$$f^* : \mathcal{D}(B\text{-Mod}) \rightarrow \mathcal{D}(A\text{-Mod})$$

induced from the forgetful functor (i.e. considering a left dg  $B$ -module as a left dg  $A$ -module via  $f$ ) and a push-forward functor

$$f_* : \mathcal{D}(A\text{-Mod}) \rightarrow \mathcal{D}(B\text{-Mod})$$

given by  $f_*(M) := B \otimes_A^{\mathbb{L}} M$ .

**Proposition 7.2.** *If  $f : (A, d_1) \rightarrow (B, d_2)$  is a quasi-isomorphism of dg algebras, then the functors  $f^*$  and  $f_*$  are inverse quasi-equivalences between  $\mathcal{D}(A\text{-Mod})$  and  $\mathcal{D}(B\text{-Mod})$ . In particular, we obtain an equivalence*

$$\overline{f^*} : \mathcal{D}_{\mathrm{sg}}(B) \xrightarrow{\cong} \mathcal{D}_{\mathrm{sg}}(A).$$

*Proof.* Note that for  $X \in \mathcal{D}(A\text{-Mod})$ , we have

$$\begin{aligned} f^* \circ f_*(X) &\cong B \otimes_A^{\mathbb{L}} X, \\ &\cong A \otimes_A B \otimes_A X \\ &\cong X. \end{aligned}$$

Similarly, for  $Y \in \mathcal{D}(B\text{-Mod})$

$$\begin{aligned} f_* \circ f^*(Y) &\cong B \otimes_A^{\mathbb{L}} Y \\ &\cong Y. \end{aligned}$$

It follows that the functors  $f^*$  and  $f_*$  are inverse quasi-equivalences. Thus  $f^*$  restricts to an equivalence between the subcategories of compact objects and also induces an equivalence between  $\mathcal{D}^b(A\text{-mod})$  and  $\mathcal{D}^b(B\text{-mod})$ , the bounded derived category of finite generated  $B$ -modules, so we have an induced equivalence of Verdier quotients

$$\overline{f^*} : \mathcal{D}_{\mathrm{sg}}(B) \xrightarrow{\cong} \mathcal{D}_{\mathrm{sg}}(A).$$

□

**Remark 7.3.** A morphism  $f : (A, d_1) \rightarrow (B, d_2)$  of dg algebras induces a morphism of dg algebras  $f \otimes f : A \otimes A^{\mathrm{op}} \rightarrow B \otimes B^{\mathrm{op}}$ . If  $f$  is a quasi-isomorphism, then so is  $f \otimes f$ . From Proposition 7.2 it follows that

$$(\overline{f \otimes f})_{\mathrm{proj}}^* : \mathcal{D}_{\mathrm{sg}}(B \otimes B^{\mathrm{op}}) \rightarrow \mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$$

is an equivalence and sends  $B$  to  $A$ . Therefore, we have an isomorphism

$$(\overline{f \otimes f})_{\mathrm{proj}}^* : \mathrm{HH}_{\mathrm{sg}}^*(B, B) \rightarrow \mathrm{HH}_{\mathrm{sg}}^*(A, A).$$



**7.2. Quasi-isomorphisms between singular Hochschild cochain complexes.** Let  $f : (A, d_1) \rightarrow (B, d_2)$  be a morphism of dg algebras. Recall that  $\Omega^p(A)$  and  $\Omega^p(B)$  are the non-commutative  $p$ -differential forms of  $A$  and  $B$ , respectively, as defined earlier and we use the identification of Lemma 3.12. Then  $f$  induces a morphism  $\Omega^p(f) : \Omega^p(A) \rightarrow \Omega^p(B)$  for  $p \in \mathbb{Z}_{\geq 0}$  given by

$$\Omega^p(f)(\overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}) := \overline{f(a_1)} \otimes \cdots \otimes \overline{f(a_p)} \otimes f(a_{p+1})$$

where we use Lemma 3.12 to identify  $\Omega^p(A)$  with  $(s\overline{A})^{\otimes p} \otimes A$ . It is clear that  $\Omega^p(f)$  is a morphism of  $A$ - $A$ -bimodules. Moreover, if  $f$  is a quasi-isomorphism, then so is  $\Omega^p(f)$ .

Now let us construct a singular Hochschild cochain complex  $\mathcal{C}_{\text{sg}}^*(A, B)$  with coefficients in  $B$ . Consider the Hochschild cochain complex  $C^*(A, \Omega^p(B))$  with coefficients in the  $A$ - $A$ -bimodule  $\Omega^p(B)$ . We define a morphism of cochain complexes

$$\theta_{A,B}^p : C^*(A, \Omega^p(B)) \rightarrow C^*(A, \Omega^{p+1}(B))$$

which sends  $\alpha \in C^{m,*}(A, \Omega^p(B))$  to the element

$$\overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}} \otimes a_{m+2} \mapsto \overline{f(a_1)} \otimes \alpha(\overline{a_2} \otimes \cdots \otimes \overline{a_{m+1}} \otimes a_{m+2}).$$

Then we define the singular Hochschild cochain complex of  $A$  with coefficients in  $B$  as

$$\mathcal{C}_{\text{sg}}^*(A, B) := \varinjlim_{p \in \mathbb{Z}_{\geq 0}} C^*(A, \Omega^p(B)).$$

with the induced differential.

Observe that, for each  $p \in \mathbb{Z}_{\geq 0}$ , there is a zig-zag of morphisms of cochain complexes

$$(19) \quad C^*(A, \Omega^p(A)) \xrightarrow{C^*(A, \Omega^p(f))} C^*(A, \Omega^p(B)) \xleftarrow{C^*(f, \Omega^p(B))} C^*(B, \Omega^p(B)).$$

These zig-zags are compatible with the inductive systems, thus we obtain a zig-zag of morphisms between singular Hochschild cochain complexes

$$\mathcal{C}_{\text{sg}}^*(A, A) \xrightarrow{\mathcal{C}_{\text{sg}}^*(A, f)} \mathcal{C}_{\text{sg}}^*(A, B) \xleftarrow{\mathcal{C}_{\text{sg}}^*(f, B)} \mathcal{C}_{\text{sg}}^*(B, B).$$

**Proposition 7.4.** *Let  $f : (A, d_1) \rightarrow (B, d_2)$  be a quasi-isomorphism of dg algebras. Then  $\mathcal{C}_{\text{sg}}^*(A, f)$  and  $\mathcal{C}_{\text{sg}}^*(f, B)$  are both quasi-isomorphisms, namely, the zig-zag above is one of quasi-isomorphisms.*

*Proof.* Note that all three complexes in the zig-zag (19) have complete decreasing filtrations with the associated quotients

$$\text{Hom}(\overline{A}^{\otimes m}, \Omega^p(A)) \longrightarrow \text{Hom}(\overline{A}^{\otimes m}, \Omega^p(B)) \longleftarrow \text{Hom}(\overline{B}^{\otimes m}, \Omega^p(B))$$

It is clear that these associated quotients are quasi-isomorphic, and thus by the usual spectral sequence argument we may conclude that the morphisms in the zig-zag in (19) are quasi-isomorphisms for each  $p \in \mathbb{Z}_{\geq 0}$ . Therefore, it follows that the morphisms  $\mathcal{C}_{\text{sg}}^*(A, f)$  and  $\mathcal{C}_{\text{sg}}^*(f, B)$  are quasi-isomorphisms.  $\square$

**Remark 7.5.** The zig-zag of chain complex quasi-isomorphisms

$$\mathcal{C}_{\text{sg}}^*(A, A) \xrightarrow{\mathcal{C}_{\text{sg}}^*(A, f)} \mathcal{C}_{\text{sg}}^*(A, B) \xleftarrow{\mathcal{C}_{\text{sg}}^*(f, B)} \mathcal{C}_{\text{sg}}^*(B, B).$$

induces an isomorphism in cohomology

$$\text{HH}_{\text{sg}}^*(f) : H^*(\mathcal{C}_{\text{sg}}^*(A, A)) \xrightarrow{H^*(\mathcal{C}_{\text{sg}}^*(A, f))} H^*(\mathcal{C}_{\text{sg}}^*(A, B)) \xrightarrow{H^*(\mathcal{C}_{\text{sg}}^*(f, B))^{-1}} H^*(\mathcal{C}_{\text{sg}}^*(B, B))$$

which are, in fact, isomorphisms of Gerstenhaber algebras. The algebra structure in the middle object is induced by the composition

$$\begin{aligned} \mathrm{HH}^m(A, \Omega^p(B)) \otimes \mathrm{HH}^n(A, \Omega^q(B)) &\rightarrow \mathrm{HH}^{m+n}(A, \Omega^p(B) \otimes_A \Omega^q(B)) \\ &\cong \mathrm{HH}^{m+n}(A, \Omega^p(A) \otimes_A \Omega^q(A)) \cong \mathrm{HH}^{m+n}(A, \Omega^{p+q}(A)) \cong \mathrm{HH}^{m+n}(A, \Omega^{p+q}(B)) \end{aligned}$$

where the first map is given by the classical Hochschild cup product construction using the  $A$ - $A$ -bimodule structure on  $\Omega^i(B)$  for  $i = p, q$  via  $f : A \rightarrow B$ , and the first and last isomorphisms are induced by the fact that  $f : A \rightarrow B$  is a quasi-isomorphism. The algebra structure on  $H^*(\mathcal{C}_{\mathrm{sg}}^*(A, A))$  and  $H^*(\mathcal{C}_{\mathrm{sg}}^*(B, B))$  is the cup product  $\cup$  defined in Section 4.1, which agrees with the classical Yoneda product  $\cup'$  as remarked in the proof of Proposition 4.1. On the other hand, it follows from [Wan2] that  $\mathrm{HH}_{\mathrm{sg}}^*(f)$  is an isomorphism of Lie algebras since the derived tensor functor  $B \otimes_A^{\mathbb{L}} -$  induces an equivalence between  $\mathcal{D}(A)$  and  $\mathcal{D}(B)$  as triangulated categories.

**7.3. The Frobenius cdga-model.** As we mentioned before, by a result of [LaSt], for any simply connected closed manifold  $M$  of dimension  $k$  there is a dg symmetric Frobenius algebra  $(A, d)$  of degree  $k$  which is simply connected (i.e.  $A^0 = \mathbb{K}$  and  $A^1 = 0$ ) (cf. Definition 2.7) together with a zig-zag of dga quasi-isomorphisms

$$(20) \quad (A, d) \xleftarrow{\cong} \cdots \xrightarrow{\cong} C^*(M)$$

such that the induced isomorphism  $H^*(A, d) \cong H^*(M)$  is one of Frobenius algebras.

When  $A$  is simply connected, the Tate-Hochschild complex  $\mathcal{D}^{*,*}(A, A)$  becomes quite simple to analyze, and thus we may compute its cohomology in terms of the Hochschild homology and cohomology of  $A$ .

**Lemma 7.6.** *Let  $A$  be a simply connected dg symmetric Frobenius algebra of degree  $k$  over a field  $\mathbb{K}$ . Then we have*

(1) *If the Euler characteristic  $\chi(A) := \mu \circ \Delta(1) = 0$ , then*

$$\mathrm{HH}_{\mathrm{sg}}^i(A, A) = \begin{cases} \mathrm{HH}^i(A, A) & \text{if } i < k-1, \\ \mathrm{HH}^{k-1}(A, A) \oplus \mathrm{HH}_0(A, A) & \text{if } i = k-1, \\ \mathrm{HH}_1(A, A) \oplus \mathrm{HH}^k(A, A) & \text{if } i = k, \\ \mathrm{HH}_{i-k+1}(A, A) & \text{if } i > k; \end{cases}$$

(2) *if the Euler characteristic  $\chi(A) \neq 0$ , then*

$$\mathrm{HH}_{\mathrm{sg}}^i(A, A) = \begin{cases} \mathrm{HH}^i(A, A) & \text{if } i \leq k-1, \\ \mathrm{HH}_{i-k+1}(A, A) & \text{if } i \geq k. \end{cases}$$

*Proof.* This is an immediate observation from the shape of the complex  $\mathcal{D}^{*,*}(A, A)$ :

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & & A^k & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & \vdots & & \vdots \\
 & & & & \vdots & & \vdots \\
 0 & \longrightarrow & \bar{A}^2 \otimes A^0 & \longrightarrow & A^2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & \longrightarrow & A^1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & \longrightarrow & A^0 & \xrightarrow{\chi} & A^k \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & \longrightarrow & A^{k-1} \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & \longrightarrow & A^{k-2} \longrightarrow \text{Hom}(\bar{A}^2, A^k) \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & \vdots & & \vdots
 \end{array}$$

We note that the non-zero terms (except  $A^k$ ) of  $\mathcal{D}^{\geq 0,*}(A, A)$  are located below the diagonal line (dotted line) and similarly the non-zero terms of  $\mathcal{D}^{< 0,*}(A, A)$  are located on or above the diagonal line. The elements on the diagonal line have the total degree  $k$ . It is clear from the diagram above that  $\text{HH}_{\text{sg}}^i(A, A)$  is isomorphic to  $\text{HH}^i(A, A)$  for  $i < k - 1$  and isomorphic to  $\text{HH}_{i-k+1}^i(A, A)$  for  $i > k$ . Let us compute  $\text{HH}_{\text{sg}}^i(A, A)$  for  $i = k - 1, k$  in the following two cases.

- (1) If the map  $\chi$  is an isomorphism (equivalently, the Euler characteristic  $\chi(A) \neq 0$ ) then both  $A^0$  and  $A^k$  are killed in the cohomology, hence  $\text{HH}_{\text{sg}}^k(A, A) \cong \text{HH}_1(A, A)$  and  $\text{HH}_{\text{sg}}^{k-1} \cong \text{HH}^{k-1}(A, A)$ .
- (2) If the map  $\chi$  is not an isomorphism, then  $\chi$  is zero since  $A^0$  and  $A^k$  are one dimensional. Then we have  $\text{HH}_{\text{sg}}^k(A, A) \cong \text{HH}_1(A, A) \oplus \text{HH}^k(A, A)$  and  $\text{HH}_{\text{sg}}^{k-1}(A, A) \cong \text{HH}^{k-1}(A, A) \oplus \text{HH}_0(A, A)$ .

□

**Remark 7.7.** If  $A$  is a Frobenius cdga model for  $M$ , then  $\chi(A) = \chi(M)$ . Indeed, the zig-zag (20) of quasi-isomorphisms of dg algebras induces an isomorphism of graded algebras between  $H^*(A)$  and  $H^*(M)$  and the pairings are compatible. Therefore  $\chi(A) = \sum_i e_i f_i = \chi(M)$ .

**Remark 7.8.** From the zig-zag (20) it follows that

$$\text{HH}_*(A, A) \cong \text{HH}_*(C^*(M), C^*(M))$$

and

$$\text{HH}^*(A, A) \cong \text{HH}^*(C^*(M), C^*(M)).$$

Recall that as shown in [Jon], there exists linear isomorphisms

$$H_*(LM) \cong \text{HH}^{k-*}(C^*(M), C^*(M))$$

and

$$H^*(LM) \cong \text{HH}_*(C^*(M), C^*(M)).$$

Now let us prove the main Theorem 7.1 of this section.

*Proof of Theorem 7.1.* From Remark 7.3, the zig-zag (20) above implies an isomorphism

$$\mathrm{HH}_{\mathrm{sg}}^*(C^*(M), C^*(M)) \cong \mathrm{HH}_{\mathrm{sg}}^*(A, A).$$

Then the result follows from Lemma 7.6, Remark 7.7 and 7.8.  $\square$

**Remark 7.9.** For a differential graded algebra  $(B, d)$ , we do not know whether there is an isomorphism between  $H^*(\mathcal{C}_{\mathrm{sg}}(B, B))$  and  $\mathrm{HH}_{\mathrm{sg}}^*(B, B)$ . But for  $C^*(M)$ , it holds by using the Frobenius cdga-model. Namely, we have the following result.

**Proposition 7.10.** *If  $M$  is a simply-connected closed manifold, then*

$$H^*(\mathcal{C}_{\mathrm{sg}}(C^*(M), C^*(M))) \cong \mathrm{HH}_{\mathrm{sg}}^*(C^*(M), C^*(M)).$$

*Proof.* Let  $(A, d)$  be a Frobenius cdga-model for  $M$ . From Proposition 7.4 it follows that

$$H^*(\mathcal{C}_{\mathrm{sg}}^*(C^*(M), C^*(M))) \cong H^*(\mathcal{C}_{\mathrm{sg}}^*(A, A)).$$

From Remark 7.3 we have  $\mathrm{HH}_{\mathrm{sg}}^*(C^*(M), C^*(M)) \cong \mathrm{HH}_{\mathrm{sg}}^*(A, A)$ . So the isomorphism

$$H^*(\mathcal{C}_{\mathrm{sg}}^*(C^*(M), C^*(M))) \cong \mathrm{HH}_{\mathrm{sg}}^*(C^*(M), C^*(M))$$

holds since  $\mathrm{HH}_{\mathrm{sg}}^*(A, A) \cong H^*(\mathcal{C}_{\mathrm{sg}}^*(A, A))$  from Corollary 3.25.  $\square$

**Example 7.11.** Let  $S^n$  be the  $n$ -dimensional sphere for  $n > 1$ . It is well-known that  $S^n$  is a *formal* manifold (cf. [DGMS]), namely, the singular cochain complex  $C^*(S^n; \mathbb{K})$  and the singular cohomology  $H^*(S^n; \mathbb{K})$  are quasi-isomorphic as dg algebras, where the latter is equipped with the trivial differential and  $\mathbb{K}$  is a field of characteristic zero. Therefore, there is an isomorphism between  $\mathrm{HH}_{\mathrm{sg}}^*(C^*(S^n), C^*(S^n))$  and  $\mathrm{HH}_{\mathrm{sg}}^*(H^*(S^n), H^*(S^n))$  from Remark 7.3. As a dg algebra,  $H^*(S^n) \cong \mathbb{K}[\epsilon_n]/(\epsilon_n^2)$ , the graded ring of dual numbers with the generator  $\epsilon_n$  in degree  $n$ . It is clear that  $A_n := \mathbb{K}[\epsilon_n]/(\epsilon_n^2)$  is a dg symmetric Frobenius algebra (with the trivial differential). Now let us compute the singular Hochschild cohomology  $\mathrm{HH}_{\mathrm{sg}}^*(A_n, A_n)$  via the Tate-Hochschild complex. We have the following two cases.

Let  $n$  be odd. The double complex  $\mathcal{D}^{*,*}(A_n, A_n)$  associated to the Tate-Hochschild complex is as follows:

$$3n\text{-row:} \quad 0 \rightarrow \epsilon_n^{\otimes 2} \otimes \mathbb{K} \rightarrow \epsilon_n \otimes \epsilon_n \rightarrow 0$$

$$2n\text{-row:} \quad 0 \longrightarrow \epsilon_n \otimes \mathbb{K} \rightarrow \epsilon_n \rightarrow 0$$

$$n\text{-row:} \quad 0 \longrightarrow \mathbb{K} \xrightarrow{\chi} \epsilon_n \longrightarrow 0$$

$$0\text{-row:} \quad 0 \rightarrow \mathbb{K} \rightarrow \mathrm{Hom}(\epsilon_n, \epsilon_n) \longrightarrow 0$$

$$-n\text{-row:} \quad 0 \rightarrow \mathrm{Hom}(\epsilon_n, \mathbb{K}) \rightarrow \mathrm{Hom}(\epsilon_n^{\otimes 2}, \epsilon_n) \rightarrow 0,$$

where  $\chi = 0$  since the Euler characteristic  $\chi(S^n)$  is zero. Notice that the vector space in each position is of dimension 1. We claim that all the horizontal differentials vanish. Indeed, for the positive row-degrees, the differentials are given by the Hochschild chains differential. Thus we have

$$d(\epsilon_n^{\otimes p} \otimes \lambda) = \epsilon_n^{\otimes p-1} \otimes \epsilon_n \lambda - (-1)^{(n-1)(n-1)(p-1)} \epsilon_n^{\otimes p-1} \otimes \lambda \epsilon_n = 0.$$

for  $d : \epsilon_n^{\otimes p} \otimes \mathbb{K} \rightarrow \epsilon_n^{\otimes p} \otimes \epsilon_n$ . Similarly, we have that the differentials for non-positive row degrees vanish. Therefore, we have the isomorphism of graded algebras  $\mathrm{HH}_{\mathrm{sg}}^*(A_n, A_n) \cong \mathbb{K}[x, x^{-1}] \otimes \Lambda[t]$ , where  $|x| = 1 - n$  and  $|t| = n$ . So we have that

$$\mathrm{HH}_{\mathrm{sg}}^*(C^*(S^n), C^*(S^n)) \cong \mathbb{K}[x, x^{-1}] \otimes \Lambda[t]$$

where  $|x| = 1 - n$ ,  $|x| = n - 1$ ,  $|t| = n$ , and  $\Lambda[t]$  denotes the exterior algebra on the generator  $t$ . Moreover, the BV-algebra structure is determined by

$$\tilde{\Delta}(x^p \otimes t) = p(x^{p-1} \otimes 1),$$

$$\tilde{\Delta}(x^p \otimes 1) = 0.$$

Let  $n$  be even. We have the analogous double complex  $\mathcal{D}^{*,*}(A_n, A_n)$ , but now  $\chi$  is an isomorphism since the Euler characteristic  $\chi(S^n) = 2$  is non-zero in  $\mathbb{K}$ . We claim that for the  $(2p+1)n$ -row ( $p \in \mathbb{Z}$ ), the (non-trivial) horizontal differential is an isomorphism. Indeed, we have

$$d(\epsilon_n^{2p} \otimes \lambda) = \epsilon_n^{\otimes 2p} \otimes \epsilon_n \lambda - (-1)^{(n-1)(n-1)(2p-1)} \epsilon_n^{\otimes 2p} \otimes \lambda \epsilon_n = 2\epsilon_n^{\otimes 2p} \otimes \lambda \epsilon_n,$$

thus the (non-trivial) differential is an isomorphism for  $p > 0$  and by the same argument we have the analogous result for  $p < 0$ . Similarly, for the  $2pn$ -row degree ( $p \in \mathbb{Z}$ ), the horizontal differentials vanish. Therefore, we have the isomorphism of graded algebras  $\mathrm{HH}_{\mathrm{sg}}^*(A_n, A_n) \cong \mathbb{K}[x, x^{-1}] \otimes \Lambda[t]$  with  $|x| = 2(n-1)$  and  $|t| = 1$ . Therefore,

$$\mathrm{HH}_{\mathrm{sg}}^*(C^*(S^n), C^*(S^n)) \cong \mathbb{K}[x, x^{-1}] \otimes \Lambda[t]$$

with  $|x| = 2(n-1)$  and  $|t| = 1$ . The BV-algebra structure is determined by

$$\tilde{\Delta}(x^p \otimes t) = (2p+1)x^p \otimes 1,$$

$$\tilde{\Delta}(x^p \otimes 1) = 0.$$

**Remark 7.12.** Our computation of the BV-algebra

$$\mathrm{HH}_{\mathrm{sg}}^{*<n}(C^*(S^n), C^*(S^n)) \cong H_{n-*}(LS^n)$$

agrees with the corresponding result [Men2] for  $n > 1$ . A BV-algebra structure was constructed in [GoHi] on  $H^*(LM, M)$  for any closed manifold  $M$ . One should be able to lift this BV-algebra structure to  $H^{>0}(LM)$  and we conjecture that it coincides with  $\mathrm{HH}_{\mathrm{sg}}^{\geq n}(C^*(M), C^*(M)) \cong H^{>0}(LM)$  in the case  $M$  is simply connected.

## APPENDIX

The following diagrams describe the product  $\star : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^*(A, A)$  for all possible cases. Blue colored output legs always represent elements of  $A$ , while black colored output legs represent elements of  $s\bar{A}$ . The degree 0 product of  $A$  is denoted by  $\mu : A \otimes A \rightarrow A$ , while the degree  $k$  coproduct is denoted by  $\Delta : A \rightarrow A \otimes A$ . A blue circle with white interior denotes the unit of the algebra  $A$  and the map  $\pi : A \rightarrow s\bar{A}$  is the natural projection. A solid black circle on the top of a single input leg in a corolla means that such an input leg is “blocked”, namely, that it cannot received any elements and thus may be ignored and the corolla may be interpreted as an element of  $(s\bar{A})^{\otimes m} \otimes A$  for some  $m$ .

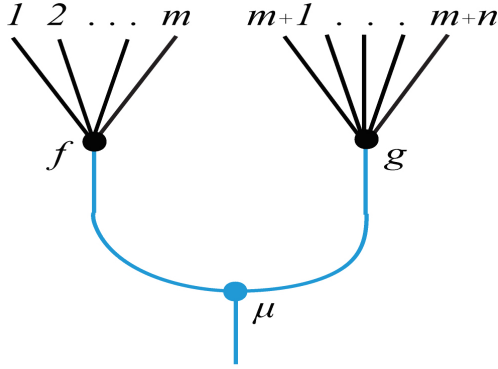


FIGURE 6. For a pair of Hochschild cochains  $f, g \in C^{*,*}(A, A)$  we have that  $f \star g$  is defined to be the classical cup product  $f \cup g$ . In the above particular example of the above picture  $m = 4$ ,  $n = 5$ , and  $f$  and  $g$  are each represented by corollas with input legs colored black and with one output leg colored blue indicating that these outputs are elements in  $A$ .

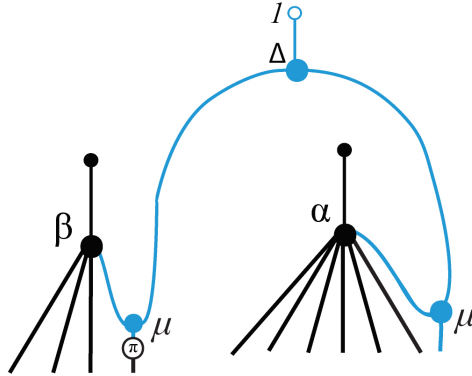


FIGURE 7. For a pair of Hochschild chains  $\alpha, \beta \in C_{*,*}(A, A)$  we have that  $\alpha \star \beta$  is defined by the diagram above. In the particular example of the above picture  $m = 6$ ,  $n = 3$ , and  $\alpha \in (s\bar{A})^{\otimes m} \otimes A$  and  $\beta \in (s\bar{A})^{\otimes n} \otimes A$  are represented by corollas with  $m + 1$  and  $n + 1$  output legs respectively and no input legs. In the literature, Hochschild chains are also represented by vertices in a circle and one of these vertices is marked; the marking corresponds to the color blue in our diagram.

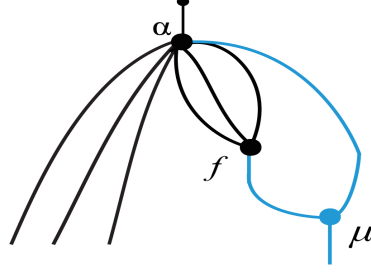


FIGURE 8. For  $\alpha \in C_{-m,*}(A, A)$  and  $f \in C^{n,*}(A, A)$  such that  $m - n \leq 0$  we have that  $f \star \alpha$  is defined to be  $f \cap \alpha$ , the classical cap product of a Hochschild cochain and a Hochschild chain. The above diagram describes such an operation. In the particular example of the above picture  $m = 6$  and  $n = 3$ . The picture for  $\alpha \star f$  is similar.

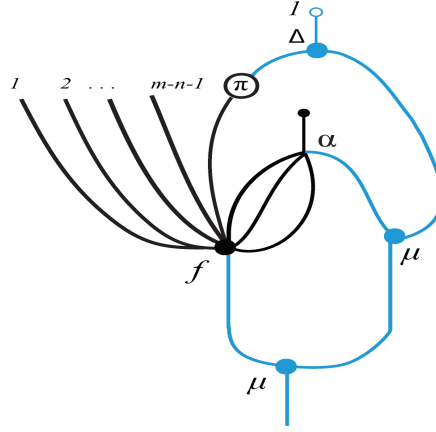


FIGURE 9. For  $\alpha \in C_{-m,*}(A, A)$  and  $f \in C^{n,*}(A, A)$  such that  $m - n > 0$  we have that  $f \star \alpha$  is defined by the diagram above. In the particular example of the above picture  $m = 3$  and  $n = 8$ . Notice that, in this case, the product  $\star$  uses the coproduct  $\Delta : A \rightarrow A \otimes A$  of degree  $k$  while the classical cap product does not use the data of the Frobenius structure of  $A$ . The diagram for  $\alpha \star f$  is similar.

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